New Vacuum Solutions for Quadratic Metric–affine Gravity

Vedad Pašić

27 August 2008
Structure of the thesis

▶ Introduction
▶ PP-waves with torsion
▶ New vacuum solutions for quadratic metric-affine gravity
▶ Discussion
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Metric–affine gravity

Spacetime considered to be a connected real 4–manifold $M$ equipped with a Lorentzian metric $g$ and an affine connection $\Gamma$, i.e.

$$\nabla_\mu u_\lambda = \partial_\mu u_\lambda + \Gamma^{\lambda}_{\mu\nu} u_\nu.$$ 

Characterisation by an independent linear connection $\Gamma$ distinguishes MAG from GR - $g$ and $\Gamma$ viewed as two totally independent quantities.

10 independent components of $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ are the unknowns of MAG.

Definition. We call a spacetime $\{M, g, \Gamma\}$ Riemannian if the connection is Levi–Civita (i.e. $\Gamma^{\lambda}_{\mu\nu} = \{\lambda\}_{\mu\nu}$), and non-Riemannian otherwise.
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Quadratic metric–affine gravity

Action is
\[ S = \int q(\mathcal{R}) \],

where \( q(\mathcal{R}) \) is a Lorentz invariant purely quadratic form on curvature.

The quadratic form \( q(\mathcal{R}) \) has 16 \( \mathcal{R}^2 \) terms with 16 real coupling constants.

Action conformally invariant, unlike Einstein–Hilbert.
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\[ \frac{\partial S}{\partial g} = 0, \tag{1} \]
\[ \frac{\partial S}{\partial \Gamma} = 0. \tag{2} \]
Known solutions for QMAG

- Einstein spaces (Yang, Mielke)
- pp-waves with parallel Ricci curvature (Vassiliev)
- Certain explicitly given torsion waves (Singh and Griffiths)
- Triplet ansatz (Hehl, MacÁas, Obukhov, Esser, ...)
- Minimal pseudoinstanton generalisation (Obukhov)
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Classical pp-waves

Definition. A pp-wave is a Riemannian spacetime which admits a non-vanishing parallel spinor field ($\nabla \chi = 0$).

Definition. A pp-wave is a Riemannian spacetime whose metric can be written locally in the form

$$ds^2 = 2dx_0 dx_3 - (dx_1)^2 - (dx_2)^2 + f(x_1, x_2, x_3)(dx_3)^2$$

in some local coordinates.

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Well known spacetimes in GR, simple formula for curvature - only trace free Ricci and Weyl pieces.
Consider the polarized Maxwell equation
\[ \ast dA = \pm i dA. \]
Plane wave solutions of this equation can be written down as
\[ A = h(\phi) m + k(\phi) l, \]
where \( \phi : M \to \mathbb{R}, \phi(x) := \int l \cdot dx. \)

Definition: A generalised pp-wave is a metric compatible spacetime with pp-metric and torsion
\[ T = \frac{1}{2} \text{Re} (A \otimes dA). \]
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Generalised pp-waves

Curvature of a generalised pp-wave is
\[ R = -\frac{1}{2} (\mathbf{l} \wedge \{\nabla\}) \otimes (\mathbf{l} \wedge \{\nabla\}) f + \frac{1}{4} \text{Re}(\mathbf{h}^2)'' (\mathbf{l} \wedge \mathbf{m}) \otimes (\mathbf{l} \wedge \mathbf{m}) \] .

Torsion of a generalised pp-wave is
\[ T = \text{Re}(\mathbf{a} \mathbf{l} + \mathbf{b} \mathbf{m}) \otimes (\mathbf{l} \wedge \mathbf{m}) , \]
where
\[ a = \frac{1}{2} \mathbf{h}'(\phi) k(\phi), \quad b = \frac{1}{2} \mathbf{h}'(\phi) h(\phi) . \]
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\[ R = -\frac{1}{2} (l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f + \frac{1}{4} \text{Re} \left( (h^2)^{''} (l \wedge m) \otimes (l \wedge m) \right). \]
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Torsion of a generalised pp-wave is
\[ T = \text{Re} \left( (a l + b m) \otimes (l \wedge m) \right), \]
where
\[ a := \frac{1}{2} h'(\varphi) k(\varphi), \quad b := \frac{1}{2} h'(\varphi) h(\varphi). \]
Main result of the thesis

Theorem

Generalised pp-waves of parallel Ricci curvature are solutions of the field equations (1) and (2).

In special local coordinates, 'parallel Ricci curvature' is written as $f_{11} + f_{22} = \text{const}$. 

Generalised pp-waves of parallel Ricci curvature admit a simple explicit description.
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Outline of the proof

Proof by 'brute force'.

We write down the field equations (1) and (2) for general metric compatible spacetimes and substitute the formulae for torsion, Ricci and Weyl into these.

Together with $\nabla \text{Ric} = 0$, we get the result.

This result was first presented in: "PP-waves with torsion and metric affine gravity", 2005 V. Pašić, D. Vassiliev, Class. Quantum Grav. 22 3961-3975.
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Interpretation

- Curvature of generalised pp-waves is split.
- Torsion and torsion generated curvature are waves traveling at the speed of light.
- Underlying pp-space can be viewed as the 'gravitational imprint' created by wave of some massless field.
- Mathematical model for neutrino?
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Mathematical model for neutrino?
Metric-affine model for a neutrino

\[ S_{\text{neutrino}} := \int \left( \xi_a \sigma_{\mu a} \dot{b} \left( \nabla_{\mu} \bar{\xi} \dot{b} \right) - \left( \nabla_{\mu} \xi_a \right) \sigma_{\mu a} \dot{b} \bar{\xi} \dot{b} \right), \]

In a generalised pp-space Weyl's equation takes form

\[ \sigma_{\mu a} \dot{b} \{ \nabla \} \mu \xi_a = 0. \]

Constructed pp-wave type solutions of Einstein-Weyl model

\[ S_{EW} := k \int R + S_{\text{neutrino}}, \]

\[ \frac{\partial S_{EW}}{\partial g} = 0, \]

\[ \frac{\partial S_{EW}}{\partial \xi} = 0. \]
Neutrino field in metric compatible spacetime described by

\[ S_{\text{neutrino}} := 2i \int \left( \xi^a \sigma_{\mu ab} (\nabla_\mu \bar{\xi}^b) - \left( \nabla_\mu \xi^a \right) \sigma^\mu_{\text{ab}} \bar{\xi}^b \right), \]
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Neutrino field in metric compatible spacetime described by

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