DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Dedicated to the memory of Djani Perčić (1975 - 2000), my friend...

Contents

1 Introduction

The aim of this paper is to explore in detail the second order linear ordinary differential equation with real periodic coefficients, also known by the name *Hill's equation*, with emphasis on stability and instability intervals and the differential operator theory connected with this problematic. These equations are used as a model in solid state physics. It is our aim to get this topic closer and explain it to a final year or master level mathematics student with previous knowledge of differential equation theory and operator theory. So some previous knowledge from courses in linear analysis, differential equation theory, functional analysis and integration and measure is required. Let us now begin with introducing some necessary notions and theory. This part of the paper is mainly going to concentrate on giving the necessary knowledge to use as reference in the latter sections of this paper, and not concentrate too much on the proofs of the results.

1.1 Floquet's theory

Let us firstly consider the known general second order differential equation

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \tag{1.1}$$

where the coefficients $a_s(x)$ $(s = 0, 1, 2)$ are complex-valued, piecewise continuous and periodic, all with period a, where a is a non-zero real constant. It is hence clear that if $\psi(x)$ is a solution of (1.1), then so is $\psi(x + a)$.

Theorem 1.1 *There exist a non-zero constant ρ and a non-trivial solution $\psi(x)$ of (1.1) such that*

$$\psi(x + a) = \rho\psi(x) \tag{1.2}$$

holds. [1]

Now let us extend this in the following theorem.

Theorem 1.2 *There are linearly independent solutions $\psi_1(x)$ and $\psi_2(x)$ of (1.1) such that either*

$$\psi_1(x) = e^{m_1 x}p_1(x), \qquad \psi_2(x) = e^{m_2 x}p_2(x),$$

where m_1 and m_2 are constants, not always distinct, and $p_1(x)$ and $p_2(x)$ are periodic with period a; or

[1] Proof of this theorem can be found in Eastham[3], section 1.1

$$\psi_1(x) = e^{mx} p_1(x), \qquad \psi_2(x) = e^{mx} (x p_1(x) + p_2(x)),$$

where m is a constant and $p_1(x)$ and $p_2(x)$ are periodic with period a. [2]

The first part of the theorem occurs when there are two linearly independent solutions of (1.1), such that (1.2) holds with either different or same values of ρ, while the second part occurs when there is only one such solution. The solutions ρ_1 and ρ_2, whether distinct or not, are called the *characteristic multipliers* of (1.1), and m_1 and m_2 from Theorem (1.2) are called the *characteristic exponents* of (1.1). The above results and their proofs are known as the *Floquet theory* after G. Floquet.

1.2 Hill's equation

Now we finally come to the Hill's equation, and in this part we explore its properties. The name of Hill's equation is given to the equation

$$\{P(x)y'(x)\}' + Q(x)y(x) = 0 \tag{1.3}$$

where $P(x)$ and $Q(x)$ are real valued and have period a. We also assume that $P(x)$ is continuous and nowhere zero and that $P'(x)$ and $Q(x)$ are piecewise continuous. Clearly, this equation is a special case of (1.1) and it is named after G.W. Hill [3]. The Hill's equation (1.3) is the equation mostly covered in this paper, so the information in this section is crucial for the continuation of our work.
We now again look at the two solutions $\psi_1(x)$ and $\psi_2(x)$ from theorem (1.2), but now we use them on equation (1.3). Let $\phi_1(x)$ and $\phi_2(x)$ be the linearly independent solutions of (1.1), which satisfy the conditions

$$\phi_1(0) = 1, \quad {\phi_1}'(0) = 0; \quad \phi_2(0) = 0, \quad {\phi_2}'(0) = 1. \tag{1.4}$$

By the proof of Theorem (1.1), we have that the characteristic multipliers ρ_1 and ρ_2 in the case of Hill's equation are solutions of the quadratic equation

$$\rho^2 - \{\phi_1(a) + {\phi_2}'(a)\}\rho + 1 = 0, \tag{1.5}$$

and hence we have that the characteristic multipliers satisfy

$$\rho_1 \rho_2 = 1. \tag{1.6}$$

[2] Proof of this theorem can be found in Eastham[3], section 1.1
[3] Go to Section 1.4 for more information on Hill

The solutions $\phi_1(x)$ and $\phi_2(x)$ of (1.3) which satisfy the boundary conditions (1.4) are real valued, by definition of Hill's equation.

Definition 1.3 *The real number D defined by*

$$D = \phi_1(a) + {\phi_2}'(a) \tag{1.7}$$

is called the discriminant of (1.3).

There are five cases we should consider in finding $\psi_1(x)$ and $\psi_2(x)$.

1. $D > 2$. Then

$$\psi_1(x) = e^{mx}p_1(x), \qquad \psi_2(x) = e^{-mx}p_2(x),$$

where $p_1(x)$ and $p_2(x)$ have period a and m is a non-zero real number, by the first part of Theorem (1.2) [4].

2. $D < -2$. Here the situation is the same as in the first case, only m must be replaced by $m + \frac{\pi i}{a}$.

3. $-2 < D < 2$. By (1.5) ρ_1 and ρ_2 are non-real and distinct. Hence by (1.6), and by the fact they are complex conjugates, there exists a real number α with $0 < a\alpha < \pi$ such that

$$e^{ia\alpha} = \rho_1, \qquad e^{-ia\alpha} = \rho_2$$

Then, by Theorem (1.2)

$$\psi_1(x) = e^{i\alpha x}p_1(x), \qquad \psi_2(x) = e^{-i\alpha x}p_2(x)$$

where $p_1(x)$ and $p_2(x)$ have period a.

4. $D = 2$. Now we have to decide which part of (1.2) we must apply, because $\rho_1 = \rho_2 = 1$, so we have to consider two cases.

   (a) $\phi_2(a) = \phi_1'(a) = 0$. A simple calculation and a manipulation of the Wronskian [5] of the matrix determined by ϕ_1 and ϕ_2, yields

   $$\psi_1(x) = p_1(x), \qquad \psi_2(x) = p_2(x)$$

   where $p_1(x)$ and $p_2(x)$ have period a. All solutions of (1.3) have period a in this case.

[4] For detailed proofs of all these five results, refer to Eastham [3], Section 1.2

[5] For the Liouville's formula for the Wronskian of two solutions of (1.1), refer to Eastham [2], section 2.3, pages 32-4

(b) $\phi_2(a)$ and $\phi'_1(a)$ are not both zero. Here

$$\psi_1(x) = p_1(x), \qquad \psi_2(x) = xp_1(x) + p_2(x)$$

where $p_1(x)$ and $p_2(x)$ have period a.

5. $D = -2$. Now $\rho_1 = \rho_2 = -1$, and again as in the previous part we have to consider two cases, depending on the part of Theorem (1.2).

(a) $\phi_2(a) = \phi'_1(a) = 0$. Doing similar manipulations to the previous part, we get that

$$\psi_1(x) = e^{\frac{\pi i x}{a}} p_1(x), \qquad \psi_2(x) = e^{\frac{\pi i x}{a}} p_2(x)$$

where $p_1(x)$ and $p_2(x)$ have period a. In this case all solutions of (1.3) satisfy

$$\psi(x + a) = -\psi(x)$$

Let us at this point also note that all functions that satisfy the above conditions are said to be *semi-periodic* with semi-period a.

(b) $\phi_2(a)$ and $\phi'_1(a)$ are not both zero. Here

$$\psi_1(x) = P_1(x), \qquad \psi_2(x) = xP_1(x) + P_2(x)$$

where $P_k(x) = e^{\frac{\pi i x}{a}} p_k(x), (k = 1, 2)$. So obviously, as above, $P_k(x)$ are also semi-periodic.

6. D non real. This is a special case that we will need later on, and D is still defined like in (1.7), only now takes complex values. In this case ρ_1 and ρ_2 are non-real and distinct, and they cannot have modulus unity, because then D would not have complex value, so there is a non-real number m with the property that re $m \neq 0$, such that

$$e^{am} = \rho_1 \qquad e^{-am} = \rho_2$$

So we obtain that

$$\psi_1(x) = e^{mx} p_1(x), \qquad \psi_2(x) = e^{-mx} p_2(x).$$

1.3 Boundedness and periodicity of solutions

We come now to the final part of this introductory section, where we are going to briefly look at the properties of solutions of the Hill's equation (1.3), with emphasis on boundedness and periodicity of solutions. Again we shall not dwell on it for too long, so for more in depth information, again please refer [3], section 1.3, because most of the proofs are omitted in this part. Now let us go through some theorems and definitions necessary for latter work.

Theorem 1.4 *1. If $|D| > 2$, all non-trivial solutions of (1.3) are unbounded in $(-\infty, \infty)$.*

2. If $|D| < 2$, all solutions of (1.3) are bounded in $(-\infty, \infty)$.

This result clearly follows from the cases 1-5 of the value of the discriminant in section 1.2.

Definition 1.5 *The equation (1.3) is said to be*

- *unstable if all non-trivial solutions are unbounded in $(-\infty, \infty)$.*
- *conditionally stable if there is a non-trivial solution which is bounded in $(-\infty, \infty)$.*
- *stable if all solutions are bounded in $(-\infty, \infty)$.*

By Theorem (1.4), (1.3) is unstable if $|D| > 2$, and stable if $|D| < 2$. Periodic and semi-periodic functions are bounded in $(-\infty, \infty)$, so from cases 4 and 5 from section 1.2, we get the following theorem.

Theorem 1.6 *The equation (1.3) has non-trivial solutions with period a if and only if $D = 2$, and with semi-period a if and only if $D = -2$. Moreover, all solutions of (1.3) have period a or semi-period a if and only if $\phi_2(a) = {\phi_1}'(a) = 0$.*

With that we finish our preliminary results, which are necessary for the next section of the paper.

1.4 Additional information

At the end of each chapter, we are going to have some additional information concerning the topic covered, which is perhaps not of our direct interest, but interesting and imortant all the same. Here we give the biography of Hill, considering the fact that we mostly cover the equation named after him, a very important mathematician. The source of all biographies in this paper is http://www-history.mcs.st-and.ac.uk/ history/ .

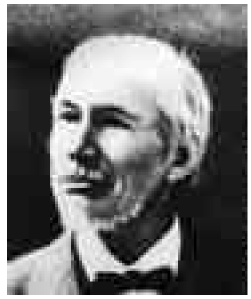

Figure 1: George William Hill (1838-1914)

1.4.1 Hill's Biography

George William Hill (Figure 1) was born on the 3rd of March 1838 in New York, USA and died on the 16th of April 1914 in West Nyack, New York, USA. After graduating from school he studied at Rutgers University graduating in 1859. The following year he began his study of the lunar theory of Delaunay and Hansen. Hill was the first to use infinite determinants to study the orbit of the Moon (1877). His Researches in Lunar Theory appeared in 1878 in the new American Journal of Mathematics. This publication contains important new ideas on the three-body problem. He also introduced infinite determinants and other methods to give increased accuracy to his results.
Newcomb persuaded Hill to develop a theory of the orbits of Jupiter and Saturn and Hill's work is a major contribution to mathematical astronomy. Hill's most important work dealt with the gravitational effects of the planets on the Moon's orbit so in this work he was considering the 4-body problem. Although he must be considered a mathematician, his mathematics was entirely based on that necessary to solve his orbits problems. He had no interest in any modern developments in other areas of mathematics. From 1898 until 1901 Hill lectured at Columbia University. Hill became a Fellow of the Royal Society (1902) receiving its Copley Medal in 1909. He was president of the American Mathematical Society from 1894 to 1896. He won the Damoiseau Prize from the Institut de France in 1898, was elected to the Royal Society of Edinburgh in 1908, elected to the academies of Belgium (1909), Christiania (1910), Sweden (1913) and others.

2 Stability and Instability Intervals

Now we come to a part of the paper which is of more interest to us in our research. As we can see, in this part we deal with *stability and instability intervals*, and how they influence our problematic. We start by extending the definitions of the previous introductory section to a more specific case.

2.1 Extending the previous information

We are still looking at Hill's equation (1.3), but in a slightly more particular form, where $Q(x)$ now has a parameter λ, such that

$$Q(x) = \lambda s(x) - q(x)$$

Here $s(x)$ and $q(x)$ are piecewise continuous with period a and $s(x)$ is bounded from below in the sense that there exists a constant $s > 0$, such that $s(x) \geq s$. Also, if we substitute $P(x)$ with $p(x)$, (1.3) now becomes

$$((p(x)y'(x))' + (\lambda s(x) - q(x))\, y(x) = 0 \tag{2.1}$$

In general, if the functions in the differential equation not only depend upon the variable x and $y(x)$, but also upon a real or complex parameter λ, then the functions $\phi_i(x)$ which form the solution will also depend upon λ. So in our case, we write $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ for the solutions of our equation (2.1) which satisfy the initial conditions (1.4) [6]. So now we define, corresponding to Definition (1.7) the discriminant

$$D(\lambda) = \phi_1(a, \lambda) + {\phi_2}'(a, \lambda) \tag{2.2}$$

Since for all λ, $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ and their derivatives with respect to x are analytic functions for fixed x, then by definition (2.2) $D(\lambda)$ is an analytic function of λ. Since $D(\lambda)$ is a continuous function of λ, the values of λ for which $|D(\lambda)| < 2$ form an open set on the real $\lambda - axis$. Since this set can be represented as a union of a countable collection of disjoint open intervals, then based on the results of Theorem (1.4), part (2), we can see that (2.1) is stable when λ is in these intervals. Similarly, when λ is in the intervals in which $|D(\lambda)| > 2$, then (2.1) is unstable. Hence, we can formulate the following definition.

Definition 2.1 • *The above described intervals which form the set of values of* λ *for which* $|D(\lambda)| < 2$ *are called the* stability intervals *of (2.1).*

[6] Refer to Eastham [2], section 1.7, page 17

- *The intervals which form the set of values of λ for which $|D(\lambda)| > 2$ are called the* instability intervals *of (2.1).*

- *The intervals which are formed by the closures of the stability intervals are called* conditional stability intervals *of (2.1)* [7].

Note that if λ is complex, then (2.1) has always unstable solutions, and at the endpoints of these intervals the solutions of (2.1) are in general unstable [8].

2.2 The eigenvalue problems

We come now to one of the most important parts of this paper, which covers the *periodic and semi-periodic eigenvalue problems*, which are one of the basic and most important problems related to (2.1). Most of this theory is heavily connected and crucial to the latter sections, especially the one related to $D(\lambda)$. We are going to be dealing here with two eigenvalue problems related to (2.1) and the interval $[0, a]$, and λ is considered as an eigenvalue parameter. Let us now tackle these two problems. But first, because they are both self adjoint eigenvalue problems, let us first define these in general.

Definition 2.2 *Let L be a second order liner differential operator*

$$L = a_0(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)$$

where x lies in a bounded closed interval $[0, a]$, each $a_r(x)$ is continuous in $[0, a]$ and $a_0(x) \neq 0$ in $[0, a]$, and let a_{ij} and b_{ij} $(1 \leq i, j \leq 2)$ be constants. Then the problem of determining a function $y(x)$ which satisfies the differential equation

$$Ly(x) = \lambda y(x)$$

in $[0, a]$, where λ is a complex parameter, and the two conditions

$$a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = 0 \tag{2.3}$$

$$a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{22}y'(b) = 0 \tag{2.4}$$

is called an eigenvalue problem*. This problem is said to be* self-adjoint *if the relation*

[7] These occur when $|D(\lambda)| \leq 2$

[8] Refer to Magnus [5], section 2.1, page 12

$$\int_0^a \overline{f_2(x)} L f_1(x) dx = \int_0^a f_1(x) \overline{L f_2(x)} dx$$

holds for all functions $f_1(x)$ and $f_2(x)$ in $C^{(2)}[0, a]$ which satisfy the above boundary conditions.

Let us now describe the two eigenvalues problems in detail.

1. The *periodic eigenvalue problem*. This problem consists of the Hill equation (2.1), which is taken to hold in $[0, a]$, and we also have the periodic boundary conditions

$$y(a) = y(0), \qquad y'(a) = y'(0) \tag{2.5}$$

This problem is a self-adjoint problem. We also know that the eigenvalues of a self-adjoint eigenvalue problem are real, so we have no problem with the complexity of λ [9]. So, we deduce that the eigenvalues form a countable set with no finite limit points, and we do this in the way of constructing the Green's function and defining a compact symmetric linear operator in an inner-product space [10]. The inner - product space we are dealing with here is that of continuous functions on $[0, a]$ with the inner product

$$< f_1, f_2 >= \int_0^a f_1(x) \overline{f_2(x)} s(x) dx$$

We shall denote the eigenfunctions by $\psi_n(x)$ and the eigenvalues by λ_n where $n = 0, 1, \ldots$ and the sequence of eigenvalues is non-decreasing and $\lambda_n \to \infty$ as $n \to \infty$. We choose $\psi_n(x)$ to be real valued and to form an orthonormal set over $[0, a]$ with weight function $s(x)$. So we have

$$\int_0^a \psi_m(x) \psi_n(x) s(x) dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2.6}$$

By (2.5), we can extend $\psi_n(x)$ to the whole $(-\infty, \infty)$ as continuously differentiable functions with period a. Hence the λ_n are the values of λ for which (2.1) has a non-trivial solution with period a.

[9] Refer to Eastham [2], Chapters 5.1-5.3, pages 84-91 for more information on self-adjoint problems

[10] More information on this in the section (2.5)

2. The *semi-periodic eigenvalue problem.* This problem consists of the Hill equation (2.1), which is taken to hold in $[0, a]$, and we also have the semi-periodic boundary conditions

$$y(a) = -y(0), \qquad y'(a) = -y'(0) \tag{2.7}$$

It is also a self-adjoint problem, but this time we shall denote the eigenfunctions by $\xi_n(x)$ and the eigenvalues by $\mu_n (n = 0, 1, \ldots)$. Again the sequence of eigenvalues is non-increasing and $\mu_n \to \infty$ as $n \to \infty$. And as before, but now by (2.7) we can extend $\xi_n(x)$ to the whole $(-\infty, \infty)$ as continuously differentiable functions with semi-period a.

From case (4) from the section 1.2 in the case of periodic eigenvalue problem we can deduce that λ_n are the zeros of the function $D(\lambda) - 2$ and that a given eigenvalue λ_n is a double eigenvalue if and only if

$$\phi_2(a, \lambda_n) = \phi_1'(a, \lambda_n) = 0$$

A similar result follows from case (5) from section 1.2 for μ_n, only this time the eigenvalues are the zeros of the function $D(\lambda) + 2$.
From now on, let $\mathcal{F}$ denote the set of all complex-valued functions $f(x)$ which are continuous in $[0, a]$ and have a piecewise continuous derivative in $[0, a]$. Let us now define the Dirichlet integral.

Definition 2.3 *Let $f(x)$ and $g(x)$ be in $\mathcal{F}$. Then the Dirichlet integral $J(f, g)$ is defined to be*

$$J(f, g) = \int_0^a \left(p(x)f'(x)\overline{g'(x)} + q(x)f(x)\overline{g(x)} \right) dx \tag{2.8}$$

If $f(x)$ and $g(x)$ satisfy the boundary conditions (2.5) and if $g(x) = \psi_n(x)$, we get that

$$J(f, \psi_n) = \lambda_n f_n \tag{2.9}$$

where f_n denotes the Fourier coefficient $\int_0^a f(x)\psi_n(x)s(x)dx$, where we have used the fact that $\psi_n(x)$ satisfies (2.1) with $\lambda = \lambda_n$. From equation (2.6) in the periodic eigenvalue problem, we can now deduce that in this case

$$J(\psi_m, \psi_n) = \begin{cases} \lambda_n & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2.10}$$

Now we come to two theorems which will finally take all of the information we have been covering in this section and give us results we can apply to concrete examples. The first one gives us a relation between the λ_n and the Dirichlet integral, in terms of an inequality.

Theorem 2.4 *Let $f(x)$ be in $\mathcal{F}$ and let it satisfy the boundary conditions (2.5). Then with the Fourier coefficients f_n defined as above, we have that*

$$\sum_{n=0}^{\infty} \lambda_n |f_n|^2 \leq J(f,f). \tag{2.11}$$

[11]

The next theorem is the last and crowning theorem of this subsection. Here 'a.e.' denotes 'almost everywhere'. In general the term is related to piecewise-continuous functions.

Theorem 2.5 *Let $\lambda_{1,n}(n \geq o)$ denote the eigenvalues in the periodic problem over $[0,a]$ denote the eigenvalues in the periodic problem over the interval $[0,a]$. In the problem we replace $p(x)$, $q(x)$ and $s(x)$ by $p_1(x)$, $q_1(x)$ and $s_1(x)$ respectively, where*

$$p_1(x) \geq p(x), \qquad q_1(x) \geq q(x), \qquad s_1(x) \leq s(x) \tag{2.12}$$

Then

(i) *if $s_1(x) = s(x)$ a.e. we have $\lambda_{1,n} \geq \lambda_n$ for all n;*

(ii) *otherwise, we have $\lambda_{1,n} \geq \lambda_n$ provided n is such that $\lambda_n \geq 0$.*

Proof. Let $\psi_{1,n}$ denote the eigenfunction corresponding to the eigenvalue $\lambda_{1,n}$ and let $J_1(f,g)$ denote the Dirichlet integral (2.8) but with $p(x)$ and $q(x)$ replaced by $p_1(x)$ and $q_1(x)$. By (2.12) we have that

$$J_1(f,f) \geq J(f,f) \tag{2.13}$$

Here we prove the theorem for the case 0. So now we consider $f(x) = \psi_{1,0}(x)$. Then by theorem (2.5) we have that

$$\lambda_{1,0} = J_1(\psi_{1,0}, \psi_{1,0}) \geq J(\psi_{1,0}, \psi_{1,0}) \geq \lambda_0 \int_0^a {\psi^2}_{1,0}(x) s(x) dx \tag{2.14}$$

Now by (2.12) we get

$$\int_0^a {\psi^2}_{1,0}(x) s(x) dx \geq \int_0^a {\psi^2}_{1,0}(x) s_1(x) dx = 1$$

[11] For proof of this theorem, please refer to [3], section 2.2, page 22

Here equality holds in the case (i) of the theorem, while in the second part of the theorem we have strict inequality. Hence, $\lambda_{1,0} \geq \lambda_0$ in the first case, but it only gives $\lambda_{1,0} \geq \lambda_0$ in the second case if $\lambda_0 \geq 0$. This proves the theorem for $n = 0$. [12].

Now finally we can use all this information for showing examples in which we can actually solve (2.1) explicitly and λ_n and μ_n can be determined.

Example 2.6 *$p(x) = s(x) = 1$, $q(x) = 0$. This is an example where (2.1) is reduced to*

$$y''(x) + \lambda y(x) = 0,$$

a well - known equation. We can show that we have $\lambda_0 = 0$, and for $m \geq 0$

$$\lambda_{2m+1} = \lambda_{2m+2} = 4\,(m+1)^2\,\frac{\pi^2}{a^2}$$

$$\mu_{2m} = \mu_{2m+1} = (2m+1)^2\,\frac{\pi^2}{a^2}$$

Example 2.7 *$p(x) = 1$, $q(x) = 0$*

$$s(x) = \begin{cases} 1 & \text{for } \left(-\frac{1}{2}a < x \leq 0\right) \\ 9 & \text{for } \left(0 < x \leq -\frac{1}{2}a\right) \end{cases}$$

In this example we just state the final results [13]. The results for the periodic eigenvalue problem are

$$\lambda_{4m+1} = 4\left(m\pi + \frac{1}{2}\alpha\right)^2/a^2, \qquad \lambda_{4m+2} = 4\left((m+1)\pi + \frac{1}{2}\alpha\right)^2/a^2,$$

$$\lambda_{4m+3} = \lambda_{4m+4} = 4(m+1)^2\pi^2/a^2$$

where $\alpha = cos^{-1}\left(\frac{7}{8}\right)$ and $0 < \alpha < \frac{1}{2}\pi$.

On the other hand, the solution for the semi-periodic eigenvalues problem is

$$\mu_{4m} = 4\left(m\pi + \frac{1}{2}\beta\right)^2/a^2, \qquad \mu_{4m+1} = 4\left(m\pi + \frac{1}{2}\gamma\right)^2/a^2,$$

$$\mu_{4m+2} = 4\left((m+1)\pi - \frac{1}{2}\gamma\right)^2/a^2, \qquad \mu_{4m+3} = 4\left((m+1)\pi - \frac{1}{2}\beta\right)^2/a^2,$$

where $\beta = cos^{-1}\left(\frac{1+\sqrt{33}}{16}\right)$ and $\gamma = cos^{-1}\left(\frac{1-\sqrt{33}}{16}\right)$ and $0 < \beta < \gamma < \pi$.

[12] For the rest of the proof please see [3], section 2.2, pages 23-25
[13] For a full solution look [3], page 25-26

2.3 The discriminant function $D(\lambda)$

In this section we are going to examine more deeply the discriminant function $D(\lambda)$, using now the knowledge we acquired in the previous section concerning the existence of eigenvalues λ_n and μ_n in the periodic and semi-periodic eigenvalue problems. The result that follows gives us much more insight into this problematic.

Theorem 2.8 (i) *The numbers λ_n and μ_n occur in the order*

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \dots \tag{2.15}$$

(ii) *In the intervals $[\lambda_{2m}, \mu_{2m}]$, $D(\lambda)$ decreases from 2 to -2.*
(iii) *In the intervals $[\mu_{2m+1}, \lambda_{2m+1}]$, $D(\lambda)$ increases from 2 to -2.*
(iv) *In the intervals $(-\infty, \lambda_0)$ and $(\lambda_{2m+1}, \lambda_{2m+2})$, $D(\lambda) > 2$.*
(v) *In the intervals (μ_{2m}, μ_{2m+1}), $D(\lambda) < 2$.* [14]

The above theorem and definition (2.1) tell us what the stability and the conditional stability intervals are. The stability intervals are (λ_{2m}, μ_{2m}) and $(\mu_{2m+1}, \lambda_{2m+1})$, because here $|D(\lambda)| < 2$, and the conditional stability intervals are the closures of these intervals. The instability intervals are $(-\infty, \lambda_0)$ together with (μ_{2m}, μ_{2m+1}) $(\lambda_{2m+1}, \lambda_{2m+2})$. However we omit any interval which is absent because $D(\lambda) \pm 2$ has a double zero. The stability intervals are never absent, nor is the *zero-th instability interval* $(-\infty, \lambda_0)$. The absence of instability intervals means that there is a value of λ for which all solutions of (2.1) have either period or semi-period a. This means that coexistence of solutions of (2.1) with period a or semi-period a occurs for that value of λ.

2.4 The Mathieu equation

In this section we study the equation in which no interval of instability can ever disappear, the Mathieu [15] equation. It is given by

$$y''(x) + (\lambda - 2q cos 2x) y(x) = 0 \tag{2.16}$$

where q is non-zero real constant. Only in the case when $q = 0$, so when this equation is reduced to the example (2.6) from the section (2.2), only the zeroth interval of instability remains. So we ignore this case. The above equation is a particular form of (2.1) where the period $a = \pi$.

[14]For proof of this theorem, look [3], section 2.3. For more information on this problematic look up [5], Chapter 2, [4], section 10.8, [1], chapter 8.3 etc.

[15]For the biography of Emile Mathieu, look at section (2.5.3)

Theorem 2.9 *For no values of λ and q, $(q \neq 0)$ do the solutions of the Mathieu equation (2.16) all have either period π or semi-period π.*

The proof of this theorem is done by contradiction.[16]. This theorem shows that coexistence does not occur for solutions of period π or semi-period π. As we have seen at the end of the last section, this implies that no instability interval of (2.16) is absent.

2.5 Additional information

2.5.1 The adjoint operator and Green's formula

In section 2.2, we left out a few formulae, so here they are for reference [17]. We are using the notation of this chapter. If $a_r(x)$ is in $C^{(2-r)}[0, a]$ where $(r = 1, 2)$, the adjoint operator exists and it is given by the formula

$$L^* = \frac{d^2}{dx^2}\overline{a_0(x)} - \frac{d}{dx}\overline{a_1(x)} + \overline{a_2(x)}$$

And in this case the Green's formula holds:

$$\int_0^a \overline{f_2(x)} L f_1(x) dx - \int_0^a f_1(x)\overline{L^* f_2(x)} dx = [f_1, f_2](b) - [f_1, f_2](a)$$

where

$$[f_1, f_2](x) = f_1'(x) a_0(x)\overline{f_2(x)} - f_1(x)\{a_0(x)\overline{f_2(x)}\}' + f_1(x) a_1(x)\overline{f_2(x)}$$

2.5.2 Infinite determinant

This topic is too not closely related to the purpose of this paper, but because it is probably the most groundbreaking work of Hill, let us just quickly mention it. Also, it gives us insight into why we were doing all that theory in the introductory chapter, and it is definitely interesting. In 1886, Hill used infinite determinants in order to investigate the characteristic values of λ in (2.1). Then in 1927, Whittaker and Watson showed that the value of Hill's infinite determinant can be expressed in terms of $D(\lambda)$. In this section we will briefly see the results of Hill and Watson, in order to understand the background of all this work [18].
We shall write the determinant in the form

$$||a_{n,m}||_k^l$$

[16] It can be found in Eastham [3], section 2.5. For more information on this problem look up [4], section 7.41, pages 177-178

[17] Also refer Eastham [2], section 5.3 and Coddington [1], Chapter 7, section 2

[18] For more thorough description of this work, refer to Magnus [5], section 2.3

where n and m vary over all of the integers between k and l. In particular, we are interested at determinants where $k = -\infty$ and $l = \infty$, or where $k = 0$ and $l = \infty$. We say that a determinant is of Hill's type if it satisfies the condition

$$\sum_{n,m} |a_{n,m} - \delta_{n,m}| < \infty$$

where $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ otherwise, and where the sum is take over all the values of n and m. We can easily see that all finite determinants are of Hill's type. The first result that we must mention is the fact that *an infinite determinant of Hill's type converges.* Since we are only interested in some of this work and not too closely, the proofs are omitted [19].

Let us now express the discriminant $D(\lambda)$ of Hill's equation (2.1) in terms of the infinite determinant. Let us firstly express (2.1) in the form

$$y'' + \left(\sum_{n=-\infty}^{\infty} g_n e^{2inx} \right) y = 0$$

where $\lambda = g_0$ and $Q(x)$ is given by the Fourier expansion $Q(x) = \sum_{n=-\infty}^{\infty} g_n e^{2inx}$. Now we can state the theorem which expresses the discriminant in terms of the Hill determinant.

Theorem 2.10 *The discriminant $D(\lambda)$ of the Hill's equation (2.1) can be expressed in two ways as an infinite determinant involving the Fourier coefficients g_n of $Q(x)$ (which are normalized so that $g_0 = 0$ and $g_{-n} = \overline{g}_n$), namely with*

$$d_0(\lambda) = \|\frac{g_{n-m}}{\lambda - 4n^2} + \delta_{n,m}\|$$

and

$$d_1(\lambda) = \|\frac{g_{n-m}}{\lambda - (2n+1)^2} + \delta_{n,m}\|$$

we have:

$$2 - D(\lambda) = 4sin^2\left(\frac{\pi}{2}\sqrt{\lambda}\right) d_0(\lambda)$$

$$2 + D(\lambda) = 4cos^2\left(\frac{\pi}{2}\sqrt{\lambda}\right) d_1(\lambda)$$

[19] Refer Magnus [5], section 2.3

2.5.3 Mathieu's Biography

Emile Mathieu was born on 15 May 1835 in Metz, France and he died on 19 Oct 1890 in Nancy, France. He is remembered especially for his discovery (in 1860 and 1873) of five sporadic simple groups named after him. These were studied in his thesis on transitive functions. Mathieu was brought up in Metz, and he attended school in that town. He excelled at school, first in classical studies showing remarkable abilities in Latin and Greek. However, once he had met mathematics when he was in his teenage years, it became the only subject which he wanted to pursue. Entering the Ecole Polytechnique in Paris his progress was almost unbelievable. It took Mathieu only eighteen months to complete the whole course and he continued to study for a doctorate. By 1859 he had been awarded his Docteur es Sciences for a thesis on transitive functions, the work which led to his initial discovery of sporadic simple groups.

Mathieu's main work, after his initial interest in pure mathematics, was in mathematical physics although he did do some important work on the hypergeometric function. From his late twenties his main efforts were devoted to the then unfashionable continuation of the great French tradition of mathematical physics, and he extended in sophistication the formation and solution of partial differential equations for a wide range of physical problems. Some of his earliest work in mathematical physics was related to his study of light and he looked at the surfaces of vibrations arising from Fresnel waves. He also worked on the polarisation of light where he highlighted some weaknesses in Cauchy's results on the topic. He worked on potential theory applied to elasticity, heat diffusion, and the vibration of bells, a very hard problem. Mathieu studied fluids, in particular examining capillary forces. He also studied magnetic induction and the three body problem where he applied his work to the perturbations of Jupiter and Saturn.

In addition to being remembered for the Mathieu groups, he is also remembered for the Mathieu functions. He discovered these functions, which are special cases of hypergeometric functions, while solving the wave equation for an elliptical membrane moving through a fluid. The Mathieu functions are solutions of the Mathieu equation.

3 Differential Operator Theory

In this final section of this paper we consider the theory of differential operators, with connection to the Hill's equation (2.1), we consider the spectral theory of self-adjoint operators and the gaps in the spectrum, and their lengths and then we finish off with the calculation of least eigenvalues. It is a section where we are going to capitalize on everything we were doing thus far.

3.1 Differential operators

In this section we are going to be looking at some applications of the theory of symmetric and self-adjoint operators in a Hilbert space to some differential operators which are in connection to the Hill's equation (2.1) from section 2. Let us denote by H the *Hilbert space* which consists of such complex valued functions, such that

$$\int_{-\infty}^{\infty} |f(x)|^2 s(x) dx$$

takes a finite value. We define the inner product on H similarly to that from section (2.2), except that we now integrate over the interval $(-\infty, \infty)$. It is given by the formula

$$< f_1, f_2 >= \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)} s(x) dx.$$

Now we can rewrite Hill's equation (2.1) in the form

$$ly(x) = \lambda y(x)$$

where l is the differential operator given by the formula

$$l = \frac{1}{s(x)} \left(-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right).$$

It is our immediate aim to show how l generates a self-adjoint operator in H. Let us now define $\mathcal{D}_0$ to be the linear manifold of functions f in H which have compact support and an absolutely continuous derivative in $(-\infty, \infty)$ such that $lf(x)$ is in the Hilbert space H. So now we can introduce a symmetric differential operator L_0 with domain $\mathcal{D}_0$ generated by l as follows

$$L_0 f(x) = lf(x) = \frac{1}{s(x)} \left(- (p(x) f'(x))' + q(x) f(x) \right).$$

Let now f and g be in $\mathcal{D}_0$. Then taking the inner product and integrating it by parts twice we get that since $p(x)$ and $q(x)$ are real-valued

$$\begin{aligned} < L_0 f, g > &= \int_{-infty}^{\infty} \left(- (p(x) f'(x))' + q(x) f(x) \right) \overline{g(x)} dx \\ &= \int_{-\infty}^{\infty} f(x) \left(- \left(p(x) \overline{g'(x)} \right)' + q(x) \overline{g(x)} \right) dx \\ &= < f, L_0 g > \end{aligned}$$

Now we need to determine what the adjoint L_0^* is, because we know it exists by the denseness of L_0 in H. We now need to introduce a second differential operator generated by l in H. Let $\mathcal{D}$ be the linear manifold of functions f in H such that $f'(x)$ exists and is absolutely continuous in $(-\infty, \infty)$ and $lf(x)$ is in H. Now define L to be the operator with domain $\mathcal{D}$ as

$$Lf(x) = lf(x).$$

Now we can formulate the theorem which finds us the adjoint operator of L_0.

Theorem 3.1 *The operator L is the adjoint of L_0.*

Let us just sketch the proof [20]. The domain ${\mathcal{D}_0}^2$ of ${L_0}^*$ consists of all the functions g in H such that

$$< L_0 f, g >=< f, g^* > \tag{3.1}$$

for all f in $\mathcal{D}_0$ and some g^* in H. Since $\mathcal{D}_0$ is dense in H, g^* is uniquely determined by g and by definition ${L_0}^* g = g^*$. In the same way as was done above, we have that $< L_0 f, g >=< f, Lg >$ for all f in $\mathcal{D}_0$ and g in $\mathcal{D}$. If we look at this result and the equation (3.1), we can see that g is in ${\mathcal{D}_0}^*$ and ${L_0}^* g = Lg$. So ${L_0}^*$ is an extension of L. The proof is completed by showing the opposite, namely that L is an extension of ${L_0}^*$. But this calculation is too lengthy to be shown, so we end the proof here.

Now that we have determined ${L_0}^*$, we can calculate the number of linearly independent functions f in ${\mathcal{D}_0}^*$ such that

$${L_0}^* f = \lambda f$$

[20] For the full proof, look at Eastham [3], section 5.1

for im $\lambda > 0$ and im $\lambda < 0$. Since ${L_0}^* = L$, the above equation gives us that

$$(p(x)f'(x))' + (\lambda s(x) - q(x))\, f(x) = 0,$$

where f is in $\mathcal{D}$. Now the above result shows us that $f''(x)$ is piecewise continuous. Hence, by the results of part (6) in section 1.2 (the one dealing with complex values of the discriminant), we get that all the non-trivial solutions $f(x)$ are not in H, so the set of linearly independent functions f in ${\mathcal{D}_0}^*$ is empty.
We can now conclude that L_0 has a unique self-adjoint extension, which is its closure $\overline{L_0}$. Then since ${L_0}^* = \overline{{L_0}^*} = \overline{L_0}$, by theorem (3.1) $L = \overline{{L_0}^*}$. Hence, L is a self-adjoint operator and it is the unique self-adjoint extension of L_0.

3.2 Gaps in the essential spectrum

Let A be a self-adjoint linear operator with domain $\mathcal{D}(A)$ in a Hilbert space $\mathcal{H}$ and let σ denote the spectrum of A. Let us now define the essential spectrum of A

Definition 3.2 *The* essential spectrum σ' *of A is defined to be the set consisting of the limit points of σ.*

An eigenvalue of infinite multiplicity is counted as limit point. By definition, σ' is a closed set on the real axis, so its complement is open and can be represented as a union of a countable number of disjoint open intervals (α_k, β_k), $(k = 0, 1, \ldots)$. These intervals are what is called *gaps* in the essential spectrum σ'. Let us now give the theorem which tells us something about the length of these gaps. In it we use the notion of *weak convergence* denoted here by $\rightarrow_w$. From functional analysis we know that $f_n \rightarrow_w f$ means that $< f_n, g > \rightarrow < f, g >$ for all g in $\mathcal{H}$. Remark that in the results that follow, if σ and σ' coincide, i.e. A has no eigenvalues with finite multiplicity, then we do not need the notion of weak convergence, and the conditions containing it may be omitted.

Theorem 3.3 *Let (f_n) be an infinite sequence in $\mathcal{D}(A)$ such that the norm $||f_n|| = 1$ for all n and $f_n \rightarrow_w 0$ as $n \to \infty$. Then*

$$\beta_k - \alpha_k \leq 2 \liminf_{n\to\infty} ||(A - \gamma_k I) f_n|| \tag{3.2}$$

where γ_k is the mid-point of (α_k, β_k).

Right away we give the next theorem which follows from the previous one.

Theorem 3.4 *A real number γ is in σ' if there is an infinite sequence (f_n) in $\mathcal{D}(A)$ such that $||f_n|| = 1$ for all n, $f_n \rightarrow_w 0$ and $||(A - \gamma I)f_n|| \to 0$ as $n \to \infty$.*

For proof of theorem (3.3) please look Eastham [3], section 5.2. The theorem (3.4) directly follows from the previous theorem and its proof.

3.3 The spectrum of L

Now we come back to our self-adjoint operator from section 3.1, and in this section we look at its spectrum σ. Let us now recall the conditional stability intervals of Hill's equation (2.1) from section 2.1, definition (2.1), and let S be the set containing all of these intervals. Let us now go onto two results directly connected to these.

Theorem 3.5 *The spectrum σ of L is purely continuous, i.e. L has no eigenvalues.*

Proof. Let us prove this quickly by contradiction. Suppose L has a real eigenvalue λ, with the corresponding eigenfunction ψ. Then we would have that $L\psi = \lambda\psi$ holds by definition of eigenvalues. So $\psi(x)$ would be a non-trivial solution of (2.1) such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 s(x) dx \tag{3.3}$$

is finite, by the properties of L. Now let us look at the possible forms of solutions of (2.1), as given by cases (1)-(5) in section 1.2. Cases (1)-(3) we discard because $\psi(x)$ cannot have modulus tending to ∞ as $x \to \infty$. So the cases remaining to consider are (4) and (5), so we would have that $\psi(x+a) = \rho\psi(x)$, where $|\rho| = 1$. Taking modulus we get that $|\psi(x+a)| = |\psi(x)|$. With this we get a contradiction, because (3.3) would not converge, i.e. would not be finite. So since there is no eigenfunction ψ, there is no eigenvalue λ, hence the spectrum σ of L is purely continuous.

Theorem 3.6 *The spectrum σ of L is identical to the set of all conditional stability intervals S.*

Proof. This is a lengthy proof, so we give here just a few main points. [21] The proof is done by showing that $S \subset \sigma$, and then the contrary. Firstly we suppose that $\gamma \in S$, and then we use the theorem (3.4) to show that $\gamma \in \sigma$ also. Again this is done by looking at cases (3)-(5) from section 1.2. Next we must define the sequence $(f_n) \in \mathcal{D}$ such that $||(f_n)|| = 1$ to use in the theorem (3.4), and as noted before we can ignore the weak convergence condition because L has no eigenvalues by the previous theorem. Next we have to prove the last condition of theorem (3.4), which is that $||(L - \gamma I)f_n|| \to 0$ as $n \to \infty$. After doing this, by theorem (3.4), we have proved that $S \subset \sigma$. To prove that $\sigma \subset S$, we suppose that μ is a real number not in S, and we have to prove that μ is also not in σ. But we do not do this here.

3.4 The lengths of the gaps in the spectrum σ

It obviously follows from the above theorem that the gaps in the spectrum σ of L are the instability intervals of the Hill's equation (2.1), defined in Definition (2.1). So theorem (3.3) can give us more information on the lengths of instability intervals.

Theorem 3.7 *Let $p(x) = s(x) = 1$. Then the length of a gap (α, β) in σ with mid-point γ satisfies the inequality*

$$\beta - \alpha \leq 2 \left(\frac{1}{a} \int_0^a (q(x) - c_0)^2 \, dx \right)^{\frac{1}{2}} \tag{3.4}$$

provided that $\gamma \geq c_0$, where

$$c_0 = \frac{1}{a} \int_0^a q(x) dx.$$

Proof. Again we just sketch the main points of the proof. First we define (f_n) similarly the proof of theorem (3.6), using as the normalization constant b_n for making $||f_n|| = 1$. So by the fact that $|f_n(x)| \leq b_n$ we now want to estimate the norm of $(L - \gamma I)f_n$ from above, and we get that

$$||(L - \gamma I)f_n|| \leq b_n \left(2n \int_0^a (q(x) - c_0(x))^2 dx \right)^{\frac{1}{2}} + o(1)$$

The theorem now follows from the fact that $b_n \sim (2na)^{\frac{1}{2}}$ as $n \to \infty$ and from theorem (3.3).

[21] The full proof can be found in Eastham [3], page 82.

Now let us consider that $q(x)$ has complex Fourier series similar to that of section 2.5.2

$$\sum c_r e^{2\pi rxi/a}$$

And hence by the Parseval formula the right hand side of (3.4) becomes

$$2\left(2\sum_{1}^{\infty}|c_r|^2\right)^{\frac{1}{2}} \tag{3.5}$$

The next theorem is an extension of (3.4) and (3.5) [22].

Theorem 3.8 *Let $p(x) = s(x) = 1$. For any integer $N \geq 1$, the length of a gap (α, β) in σ with mid-point γ satisfies*

$$\beta - \alpha \leq 2\left(2\sum_{N+1}^{\infty}|c_r|^2\right)^{\frac{1}{2}} + \frac{2\pi}{a}\left(\frac{2\sum_1^N r^2|c_r|^2}{\gamma - c_0 - 2\sum_1^N |c_r|}\right)^{\frac{1}{2}} \tag{3.6}$$

provided

$$\gamma > c_0 + 2\sum_{1}^{N}|c_r| \tag{3.7}$$

We now have two corollaries of this.

Corollary 3.9 *Let $p(x) = s(x) = 1$. Then the length of a gap in σ goes to zero as the gap recedes to $+\infty$.*

Proof. Let $\varepsilon > 0$ be arbitrary. In inequality (3.6), we define N to be such that the first term on the right is not greater than $\frac{1}{2}\varepsilon$. Now let us define $\gamma_0(\varepsilon)$ to be such that when $\gamma = \gamma_0(\varepsilon)$, (3.7) holds and the second term on the right of (3.6) is not greater than $\frac{1}{2}\varepsilon$. Hence (3.6) gives $\beta - \alpha \leq 2\varepsilon$ for $\gamma \geq \gamma_0(\varepsilon)$. Hence the result follows.

Corollary 3.10 *Let $p(x) = s(x) = 1$. Now let $q(x)$ be absolutely continuous with $q'(x)$ in $L^2(0, a)$. Then we have that*

$$\beta - \alpha \leq \frac{2\pi}{a}\left(\frac{2\sum_1^\infty r^2|c_r|^2}{\gamma - c_0 - 2\sum_1^\infty |c_r|}\right)^{\frac{1}{2}} \tag{3.8}$$

with the condition that (3.7) holds with $N = \infty$.

Proof. By the conditions on $q(x)$ the result follows from (3.6) by letting $N \to \infty$.

[22] For proof of theorem (3.8), look up Eastham [3], section 5.4

3.5 The least eigenvalues

Now in this section we come to the last part of our coverage of the application of differential operator theory to the study of the equation (2.1). We now go back to section 2.2 where we dealt with eigenvalue problems of Hill's equation, and again as before denote by $\mathcal{F}$ the set of all complex-valued functions $f(x)$ which are continuous in $[0, a]$ and have a piecewise continuous derivative in $[0, a]$. Now also recall the Dirichlet integral $J(f, g)$ defined in equation (2.8) for f and g in $\mathcal{F}$. From the proof of theorem (2.4) [23], we get that λ_0 is equal to

$$min \left(\frac{J(f, f)}{\int_0^a |f(x)|^2 s(x) dx} \right) \tag{3.9}$$

where the minimum is taken over all non-trivial $f(x)$ in $\mathcal{F}$ which satisfy the periodic boundary condition (2.5). Similarly, (3.9) is equal to μ_0 if $f(x)$ satisfies the semi-periodic boundary condition (2.7) instead of (2.5). The minimum is attained when $f(x)$ equals the corresponding eigenfunction $\psi_0(x)$ for the periodic case, i.e. $\xi_0(x)$ for the semi-periodic case. Since these eigenfunctions are real functions, in these cases we can restrict $f(x)$ to the real values. Let us now consider estimating these eigenvalues.

Theorem 3.11 *Let us define constants c and M in the following way*

$$c = \frac{\int_0^a q(x) dx}{\int_0^a s(x) dx}, \qquad M = \inf p(xs(x). \tag{3.10}$$

Then we can estimate λ_0 both from below and above in the following way

$$c - \frac{1}{16M} \left(\int_0^a |q(x) - cs(x)| dx \right)^2 \leq \lambda_0 \leq c. \tag{3.11}$$

[24]

Theorem 3.12 *Let q_0 and q_1 denote the cosine coefficients*

$$q_r = \frac{1}{a} \int_0^a q(x) cos(2r\pi x/a) dx \qquad (r = 0, 1)$$

and similarly for p_0, p_1, s_0 and s_1. Then we have that

$$\mu_0 \leq \frac{\left(\pi^2 \frac{p_0 \mp p_1}{a^2} + q_0 \pm q_1\right)}{s_0 \pm s_1} \tag{3.12}$$

where we read all the upper and lower minus and plus signs together.

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[23]Eastham [3], section 2.2, page 23

[24] For proof of this theorem look up Eastham [3], section 5.5

Proof. Using (3.9) by taking $f(x) = cos(\pi x/a)$ applied to μ_0 in the semi-periodic case. Hence we get

$$\mu_0 \le \frac{\frac{\pi^2}{a^2}\int_0^a p(x)sin^2\left(\frac{\pi x}{a}\right)dx + \int_0^a q(x)cos^2\left(\frac{\pi x}{a}\right)dx}{\int_0^a s(x)cos^2\left(\frac{\pi x}{a}\right)dx}$$

Now if we express $sin^2(\pi x/a)$ and $cos^2(\pi x/a)$ in terms of $cos(2\pi x/a)$, we get the required result with with the upper signs. To obtain it with lower signs, we just take $f(x) = sin(\pi x/a)$.

3.6 Additional information

In this last additional information section we show how this theory can be taken out of the context of ordinary differential equations, and applied to the multidimensional case, using Schrodinger's equation. Although Erwin Schrodinger did not play a great role in the matter we covered, he was such an exeptional scientist that his biography deserves to appear in this section as well.

3.6.1 The periodic Schrodinger equation

This is a topic where we talk about the *Schrodinger equation* [25], and show its connection with the theory of Hill's equation (2.1) from previous chapters. However, we do not delve deeply into this problematic, but we mention it for its importance. We define the Schrodinger equation by

$$\Delta\psi(\mathbf{x}) + (\lambda - q(\mathbf{x}))\,\psi(\mathbf{x}) = 0 \tag{3.13}$$

holding in the whole N - dimensional space E_N $(N > 1)$. Here $\mathbf{x}$ denotes a vector in E_N, Δ is the Laplace operator, λ a real parameter and $q(\mathbf{x})$ is real-valued and periodic. What this means in an N -dimensional space is that there are N linearly independent vectors $\mathbf{a}_j$ in E_N such that $q(\mathbf{x}+\mathbf{a}_j) = q(\mathbf{x})$. The Laplace operator is defined in the standard way

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \ldots + \frac{\partial^2}{\partial {x_N}^2}.$$

Now a large amount of results concerning equation (2.1) depend upon Floquet theory and the investigation of the discriminant $D(\lambda)$ in sections 1

[25] For more information on Scrodinger, see section 3.6.2

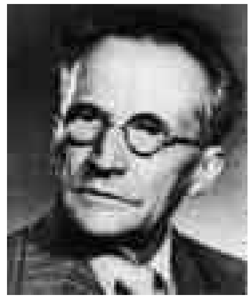

Figure 2: E. R. J. Schrodinger (1887-1961)

and 2, i.e. rely on investigation of ordinary differential equations. In a multi-dimensional case we deal with partial differential equations. However, quite a few results for equation (2.1) can be proved by using different methods, which do not use Floquet theory and $D(\lambda)$ which apply to equation (3.13) as well. Most important of these are the eigenvalue problems, periodic and k - periodic, the variational results based on the Dirichlet integral, most of the differential operator theory, including self-adjoint problem and the spectral theory, and also the existence of conditional stability intervals for (3.13) can be established. However, this is beyond the scope of this paper, and we leave it at that [26].

3.6.2 Schrodinger's biography

Erwin Rudolf Josef Alexander Schrodinger (Figure 2) was born on 12th of August 1887 in Erdberg, Vienna, Austria and he died on 4th January 1961 in Vienna, Austria. Schrodinger entered the University of Vienna in 1906. On 20 May 1910, Schrodinger was awarded his doctorate with a doctoral dissertation 'On the conduction of electricity on the surface of insulators in moist air'. Then he was appointed to an assistantship at Vienna but, rather surprisingly, in experimental physics rather than theoretical physics. In 1914 Schrodinger's first important paper was published developing ideas of Boltzmann. However, with the outbreak of World War I, Schrodinger received orders to take up duty on the Italian border. In the spring of 1917 Schrodinger was sent back to Vienna, assigned to teach a course in meteorology. He was able to continue research and he published his first results on quantum theory. After the end of the war he continued working at Vienna. From 1918 to 1920 he made substantial contributions to colour theory. Schrodinger had worked at Vienna on radioactivity, proving the statistical

[26] Refer to Eastham [3], section 6 for coverage of the periodic Schrodinger equation

nature of radioactive decay. He had also made important contributions to the kinetic theory of solids, studying the dynamics of crystal lattices.

From 1921 he studied atomic structure. Then in 1924 he began to study quantum statistics, and soon after this he read de Broglie's thesis which was to have a major influence on his thinking. Schrodinger published his revolutionary work relating to wave mechanics and the general theory of relativity in a series of six papers in 1926. Wave mechanics, as proposed by Schrodinger in these papers, was the second formulation of quantum theory, the first being matrix mechanics due to Heisenberg. He took up the post of Planck's chair in Berlin on 1 October 1927 and there he became a colleague of Einstein's. Schrodinger decided in 1933 that he could not live in a country in which persecution of Jews had become a national policy. On 4 November 1933 Schrodinger arrived in Oxford.

Soon after he arrived in Oxford, Schrodinger heard that, for his work on wave mechanics, he had been awarded the Nobel prize. In 1935 Schrodinger published a three-part essay on 'The present situation in quantum mechanics' in which his famous Schrodinger's cat paradox appears. This was a thought experiment where a cat in a closed box either lived or died according to whether a quantum event occurred. The paradox was that both universes, one with a dead cat and one with a live one, seemed to exist in parallel until an observer opened the box.

He received an offer from the University of Graz and he went to Austria and spent the years 1936-1938 in Graz. After the Anschluss the Germans occupied Graz and renamed the university Adolf Hitler University. The Nazis could not forget the insult he had caused them when he fled from Berlin in 1933 and on 26 August 1938 he was dismissed from his post for 'political unreliability'. He fled quickly to Rome from where he wrote to de Valera as President of the League of Nations. De Valera offered to arrange a job for him in Dublin in the new Institute for Advanced Studies he was trying to set up. From Rome Schrodinger went back to Oxford, and there he received an offer of a one year visiting professorship at the University of Gent. After his time in Gent, Schrodinger went to Dublin in the autumn of 1939. There he studied electromagnetic theory and relativity and began to publish on a unified field theory. His first paper on this topic was written in 1943. He remained in Dublin until he retired in 1956 when he returned to Vienna and wrote his last book Meine Weltansicht (1961) expressing his own metaphysical outlook. During his last few years Schrodinger remained interested in mathematical physics and continued to work on general relativity, unified field theory and meson physics.

3.6.3 Internet availability an some final words

A copy of this dissertation can be downloaded from the website http://www.maths.vedad.cjb.net directly. Apart from this I will continue to put some additional information concerning this area of mathematics, so come back soon to this website! With this final note, we finish this paper and hope that it gave you enough information on this topic, and that you found it useful. This is a very interesing and useful part of mathematics, so I hope reading about it was as interesting for you as it was for me. Until next time.

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