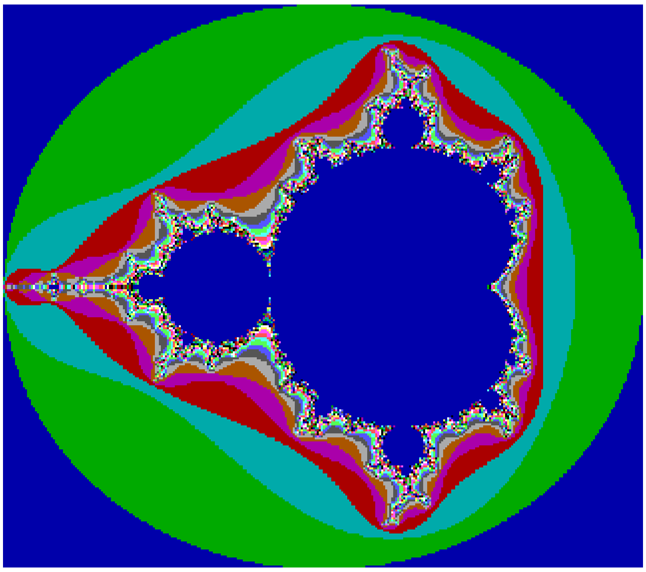
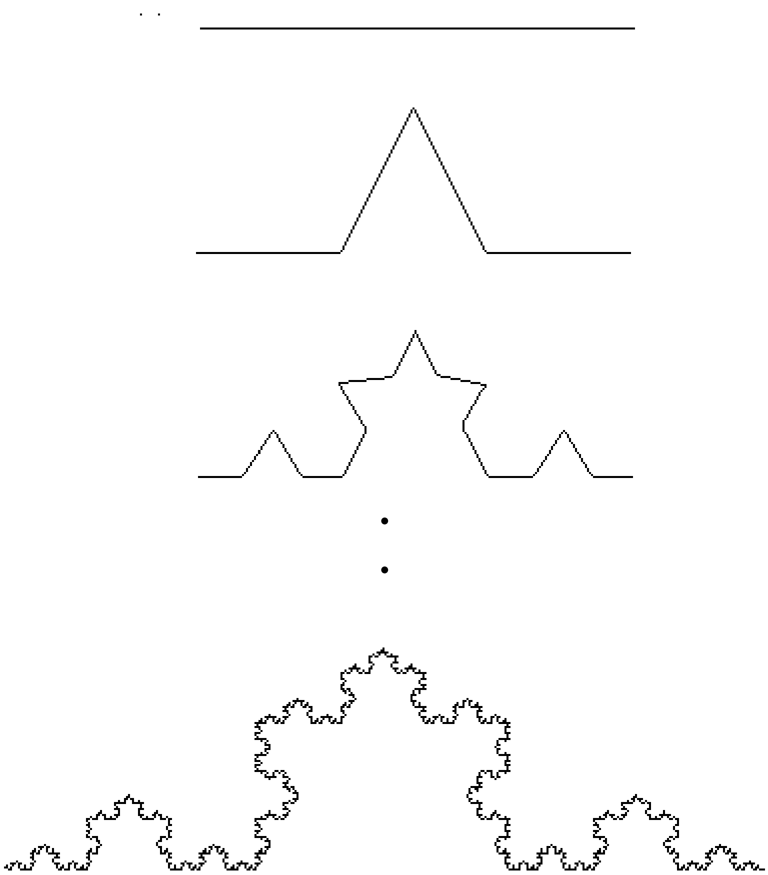
Mathematics Seminar Project

FRACTAL DIMENSION



By: Vedad Pašić
Supervisor: Professor Dmitri G. Vassiliev

THE VON KOCH CURVE



- Made up of four copies of itself scaled by a factor 1/3, and it has dimension

  $d = -\ln 4/\ln (1/3) = \ln 4/\ln 3 = 1.262$;

- *Similarity dimension* of the set *:* a set made up of *m* copies of itself scaled by a factor *r* might be thought of as having dimension $d = -\ln m/\ln r$.

<u>WHAT IS A FRACTAL</u>

We consider a set *F* in Euclidean space to be fractal if it has all or most of the following properties:

- F has a fine structure, i.e. detail on arbitrarily small scales;
- F is too irregular to be described in traditional geometrical language, both locally and globally;
- Often F has some form of self – similarity, perhaps approximate or statistical;
- Usually, the “fractal dimension” of F (defined in some way) is greater than its topological dimension;
- In many cases of interest F has a very simple, perhaps recursive definition;
- Often F has a natural appearance.

Hausdorff measure and dimension

Felix Hausdorff (1869 – 1942)

- The oldest definition of *fractal dimension*;

- Defined for any set;

- Hausdorff *measure* $\boldsymbol{H}^s(F)$ in 3^n proportional to the Lebesgue measure (the n-dimensional volume);

- The Hausdorff *dimension* defined by:

  $$\dim_H F = \inf\{s: \boldsymbol{H}^s(F) = 0\} = \sup\{s: \boldsymbol{H}^s(F) = \infty\}$$

  is the value s at which $\boldsymbol{H}^s(F)$ ‘jumps’ from ∞ to 0.

EXAMPLE OF HAUSDORFF DIMENSION

Construction of the middle third Cantor set F

Let F be the middle third Cantor set. If s=ln 2 / ln 3 = 0.6309... then $\dim_H F = s$ *and* $\frac{1}{2} \leq \mathcal{H}^s(F) \leq 1$.

Minkowski dimension

Hermann Minkowski (1864 – 1909)

- A different definition of dimension which is more applicable in calculating the dimension of a set *F;*
- Several different versions of this defintion :

Let F be any non-empty bounded set of 3^n. *The Minkowski dimension of F is given by:*

$$\dim_M F = \lim_{\varepsilon \to 0} \frac{\ln N_\varepsilon(F)}{-\ln \varepsilon}$$

- Recall that the ε - *neighbourhood* F_ε of *F* is

$$F_\varepsilon = \{ x \in 3^n : dist\,(x, F) < \varepsilon \},$$

where *dist (x, F)* is the Euclidean distance in 3^n. Then, if F is a subset of 3^n,

$$\dim_M F = n - \lim_{\varepsilon \to 0} \frac{\ln vol^n(F_\varepsilon)}{\ln \varepsilon}$$

- A very important relation between the Minkowski and Hausdorff dimension:

$$\dim_{\mathrm{H}} \mathrm{F} \leq \underline{\dim}_M F \leq \overline{\dim}_M F$$

- A very important form of the Minkowski dimension is the *interior Minkowski dimension of the boundary*

$$\dim_I \partial\Omega = n - \lim_{\varepsilon \to 0} \frac{\ln vol^n \partial\Omega_\varepsilon^i}{\ln \varepsilon}$$

where $\partial\Omega_\varepsilon^i = \{ \underline{x} \in \Omega : \mathrm{dist}(\underline{x}, \partial\Omega) < \varepsilon \}$ is the *interior* ε *- neighbourhood of* the boundary $\partial\Omega$ of a bounded open set Ω.

- The notion of the *'interior Minkowski dimension'* is connected with the problem of the *eigenvalue counting function*,

$$N(\lambda) = \# \{ \lambda_j (\Omega) < \lambda \}$$