# New Vacuum Solutions for Quadratic Metric–affine Gravity

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17th September 2009

Ohrid, Macedonia

# Structure of talk

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- Mathematical model

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- Current and future work

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The quadratic form $q(R)$ has 16 $R^2$ terms with 16 real coupling constants.

Action conformally invariant, unlike Einstein–Hilbert.

# Field equations

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Independent variation of $g$ and $\Gamma$ produces the system of Euler–Lagrange equations

$$\partial S/\partial g = 0, \quad (1)$$

$$\partial S/\partial \Gamma = 0. \quad (2)$$

# Known solutions for QMAG

**Definition.** We call a spacetime $\{M, g, \Gamma\}$ *Riemannian* if the connection is Levi–Civita (i.e. $\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$).

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- Minimal pseudoinstanton generalisation (Obukhov).

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$$ds^2 = 2\,dx^0\,dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3)\,(dx^3)^2$$

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Well known spacetimes in GR, simple formula for curvature - only trace free Ricci and Weyl pieces.

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$$T := \frac{1}{2}\mathrm{Re}(A \otimes dA).$$

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- Curvature of a generalised pp-wave is

$$R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f + \frac{1}{4}\mathrm{Re}\left( (h^2)'' \, (l \wedge m) \otimes (l \wedge m) \right).$$

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- Torsion of a generalised pp-wave is

$$T = \mathrm{Re}\left((a\,l + b\,m) \otimes (l \wedge m)\right),$$

where

$$a := \frac{1}{2}h'(\varphi)\;k(\varphi), \quad b := \frac{1}{2}h'(\varphi)\;h(\varphi).$$

# Main result of the thesis

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**Theorem** Generalised pp-waves of parallel Ricci curvature are solutions of the field equations (1) and (2).

In special local coordinates, ‘parallel Ricci curvature’ is written as $f_{11} + f_{22} = \mathrm{const}$.

Generalised pp-waves of parallel Ricci curvature admit a simple explicit description.

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- Proof by 'brute force'.
- We write down the field equations (1) and (2) for general metric compatible spacetimes and substitute the formulae for torsion, Ricci and Weyl into these.
- Together with $\nabla Ric = 0$, we get the result.
- This result was first presented in :
  "PP-waves with torsion and metric affine gravity", V. Pasic, D. Vassiliev, *Class. Quantum Grav. 22 3961-3975*.

# Physical interpretation?

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- Curvature of generalised pp-waves is split.
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- Underlying pp-space can be viewed as the ‘gravitational imprint’ created by wave of some massless field.
- Mathematical model for some massless particle?

# Metric-affine vs Einstein-Weyl

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Look at Weyl action

$$S_W := 2i \int \left( \xi^a \, \sigma^\mu{}_{a\dot{b}} \, (\nabla_\mu \bar{\xi}^{\dot{b}}) \; - \; (\nabla_\mu \xi^a) \, \sigma^\mu{}_{a\dot{b}} \, \bar{\xi}^{\dot{b}} \right),$$

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There exist pp-wave type solutions of Einstein-Weyl model

$$S_{\mathrm{EW}} := k \int \mathcal{R} + S_{\mathrm{neutrino}},$$

$$\partial S_{\mathrm{EW}} / \partial g = 0,$$

$$\partial S_{\mathrm{EW}} / \partial \xi = 0.$$

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- Extension of Singh's work (*Phys. Let. A* **145** 7, *Class. Quantum Grav.* **7** 2125) in the Yang–Mills case to the general?
- Future collaboration with Vassiliev: teleparallelism, (massless) Dirac equation and Cosserat elasticity (alternative model for electron?), etc.

Thank You!