Structure of presentation

- Metric – affine gravity: The results of previous joint work.
- Physical interpretation of these results.
- Future work for my PhD: The spectrum of the massless Dirac operator.
- Discussion.
Alternative theory of gravity.

Natural generalization of Einstein’s GR, which is based on a spacetime with Riemannian metric $g$ of Lorentzian signature.

We consider spacetime to be a connected real 4-manifold $M$ equipped with Lorentzian metric $g$ and an affine connection $\Gamma$.

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The 10 independent components of the symmetric metric tensor $g_{\mu\nu}$ and 64 connection coefficients $\Gamma^\lambda_{\mu\nu}$ are unknowns of MAG.
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We define our action as

\[ S := \int q(R) \]  \hspace{1cm} (1)\]

where \( q(R) \) is a quadratic form on curvature \( R \).

The system of Euler–Lagrange equations:

\[ \frac{\partial S}{\partial g} = 0, \]  \hspace{1cm} (2)\]

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We write down explicitly our field equations (2), (3) under following assumptions:

(i) our spacetime is metric compatible,
(ii) curvature has symmetries

\[ R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \quad \varepsilon_{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0, \]

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The main result is

**Lemma**

*Under the above assumptions (i) – (iii), the field equations (2), (3) are*

\[
0 = d_1 \mathcal{W}^{\kappa\lambda\mu\nu} \text{Ric}_{\kappa\mu} + d_3 \left( \text{Ric}^\lambda_{\kappa\nu} \text{Ric}_{\kappa\mu} - \frac{1}{4} g_{\lambda\nu} \text{Ric}_{\kappa\mu} \text{Ric}^{\kappa\mu} \right) \tag{4}
\]
Main result so far

The main result is

Lemma

Under the above assumptions (i) – (iii), the field equations (2), (3) are

\[ 0 = d_1 \mathcal{V}^{\kappa\lambda\mu\nu} R_{\kappa\mu} + d_3 \left( R_{\lambda}^{\kappa} R_{\kappa}^{\nu} - \frac{1}{4} g^{\lambda\nu} R_{\kappa\mu} R_{\kappa\mu} \right) \]
New representation of the field equations

\[ 0 = d_6 \nabla_\lambda \text{Ric}_{\kappa \mu} - d_7 \nabla_\kappa \text{Ric}_\lambda \mu \\
+ d_6 \left( \text{Ric}_\kappa \eta (K_{\mu \eta \lambda} - K_{\mu \lambda \eta}) + \frac{1}{2} g_{\lambda \mu} \mathcal{W}^{\eta \zeta}_{\kappa \xi} (K^{\xi}_{\eta \zeta} - K^{\xi}_{\zeta \eta}) + \frac{1}{2} g_{\mu \lambda} \text{Ric}_\xi \eta K^{\xi}_{\eta \kappa} \\
+ g_{\mu \lambda} \text{Ric}_\kappa \eta K^{\xi}_{\xi \eta} - K^{\xi}_{\xi \lambda} \text{Ric}_{\kappa \mu} + \frac{1}{2} g_{\mu \kappa} \text{Ric}_\xi \xi (K^{\eta}_{\xi \eta} - K^{\eta}_{\eta \xi}) \right) \\
- d_7 \left( \text{Ric}_\lambda \eta (K_{\mu \eta \kappa} - K_{\mu \kappa \eta}) + \frac{1}{2} g_{\kappa \mu} \mathcal{W}^{\kappa \zeta}_{\lambda \xi} (K^{\xi}_{\eta \zeta} - K^{\xi}_{\zeta \eta}) + \frac{1}{2} g_{\mu \kappa} \text{Ric}_\xi \eta K^{\xi}_{\eta \lambda} \\
+ g_{\kappa \mu} \text{Ric}_\lambda \eta K^{\xi}_{\xi \eta} - K^{\xi}_{\xi \kappa} \text{Ric}_\lambda \mu + \frac{1}{2} g_{\mu \kappa} \text{Ric}_\lambda \xi (K^{\eta}_{\xi \eta} - K^{\eta}_{\eta \xi}) \right) \\
+ b_{10} \left( g_{\mu \lambda} \mathcal{W}^{\eta \zeta}_{\kappa \xi} (K^{\xi}_{\zeta \eta} - K^{\xi}_{\eta \zeta}) + g_{\mu \kappa} \mathcal{W}^{\eta \zeta}_{\lambda \xi} (K^{\xi}_{\eta \zeta} - K^{\xi}_{\zeta \eta}) \\
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+ g_{\kappa \mu} \text{Ric}_\lambda \eta K^{\xi}_{\xi \eta} - g_{\lambda \mu} \text{Ric}_\kappa \eta K^{\xi}_{\xi \eta} + \text{Ric}_{\mu \kappa} K^{\eta}_{\lambda \eta} - \text{Ric}_{\mu \lambda} K^{\eta}_{\kappa \eta} \right) \\
+ 2b_{10} \left( \mathcal{W}^{\eta \zeta}_{\mu \kappa \xi} (K^{\xi}_{\eta \lambda} - K^{\xi}_{\lambda \eta}) + \mathcal{W}^{\eta \zeta}_{\mu \lambda \xi} (K^{\xi}_{\kappa \eta} - K^{\xi}_{\eta \kappa}) \\
- K_{\mu \xi \eta} \mathcal{W}^{\eta \zeta}_{\kappa \lambda} - K^{\xi}_{\xi \eta} \mathcal{W}^{\eta \zeta}_{\mu \lambda \kappa} \right) \]

where \( d_1, d_3, d_6, d_7, b_{10} \) are some real constants.
The first task

We are going to try to generalize pp–waves as follows

**Conjecture**

*There exists a new class of spacetimes with pp–metric and purely axial torsion which are solutions of the field equations (2), (3).*

**Expectations:**

- to prove or disprove conjecture above.
- to give a physical interpretation of the new solutions and compare them with existing Riemannian solutions.
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Massless Dirac action:

\[ S_{\text{neutrino}} := 2i \int \left( \xi^a \sigma^\mu_{ab} (\nabla_\mu \bar{\xi}^b) - (\nabla_\mu \xi^a) \sigma^\mu_{ab} \bar{\xi}^b \right). \]

In Einstein–Weyl theory the action is given by:

\[ S_{\text{EW}} = S_{\text{neutrino}} + k \int \mathcal{R}. \]

We obtain the well known Einstein–Weyl field equations

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Physical interpretation

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\[ \frac{\partial S_{\text{EW}}}{\partial g} = 0, \quad \frac{\partial S_{\text{EW}}}{\partial \xi} = 0. \]
The massless Dirac operator is the matrix operator

\[ W = -i \sigma^\alpha \left( \frac{\partial}{\partial x^\alpha} + \frac{1}{4} \sigma_\beta \left( \frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \beta \atop \alpha \gamma \right\} \sigma^\gamma \right) \right). \]  

The massless Dirac operator (8) describes a single massless neutrino living in 3-dimensional compact universe \( M \).

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Let $M$ be a 3-dimensional connected oriented manifold equipped with a Riemannian metric $g_{\alpha\beta}$ and let $W$ be the corresponding massless Dirac operator (8).

Two basic examples when the spectrum of $W$ can be calculated explicitly:

- the unit torus $\mathbb{T}^3$ equipped with Euclidean metric.
- the unit sphere $S^3$ equipped with metric induced by the natural embedding of $S^3$ in Euclidean space $\mathbb{R}^4$.

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Physically, this means that in these two examples there is no difference between the properties of the particle (massless neutrino) and antiparticle (massless antineutrino).

For a general oriented Riemannian 3-manifold there is no reason for the spectrum of massless Dirac operator $W$ to be symmetric (M. F. Atiyah, V. K. Patodi and I. M. Singer).

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Pfäffle: the example based on the idea of choosing a 3-manifold with flat metric but highly nontrivial topology!

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The plan

Try to achieve the spectral asymmetry of the massless Dirac operator (8) on the unit sphere $S^3$.

The ideas: the metric which depends on small parameter $\varepsilon$, Hopf fibration.
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Welcome to Tuzla!