# Metric-affine vs Spectral Theoretic Characterization of the Massless Dirac Operator

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# Structure of presentation

- Metric – affine gravity: The results of previous joint work.
- Physical interpretation of these results.
- Future work for my PhD: The spectrum of the massless Dirac operator.
- Discussion.

# Metric – affine gravity

Alternative theory of gravity.

Natural generalization of Einstein's GR, which is based on a spacetime with Riemannian metric $g$ of Lorentzian signature.

We consider spacetime to be a connected real 4-manifold $M$ equipped with Lorentzian metric $g$ and an affine connection $\Gamma$.

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The 10 independent components of the symmetric metric tensor $g_{\mu\nu}$ and 64 connection coefficients $\Gamma^{\lambda}{}_{\mu\nu}$ are unknowns of MAG.

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We define our action as

$$S := \int q(R) \tag{1}$$

where $q(R)$ is a quadratic form on curvature $R$.

The system of Euler – Lagrange equations:

$$\frac{\partial S}{\partial g} = 0, \tag{2}$$

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# New representation of the field equations

We write down explicitly our field equations (2), (3) under following assumptions:

(*i*) our spacetime is metric compatible,

(*ii*) curvature has symmetries

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \qquad \varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0,$$

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**Lemma**

*Under the above assumptions $(i)-(iii)$, the field equations (2), (3) are*

$$0 = d_1 \mathcal{W}^{\kappa\lambda\mu\nu} Ric_{\kappa\mu} + d_3 \left( Ric^{\lambda\kappa} Ric_{\kappa}{}^{\nu} - \frac{1}{4} g^{\lambda\nu} Ric_{\kappa\mu} Ric^{\kappa\mu} \right) \quad (4)$$

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\begin{aligned}
0 \quad = \quad & d_6 \nabla_\lambda Ric_{\kappa\mu} - d_7 \nabla_\kappa Ric_{\lambda\mu} \\
+ \quad & d_6 \Big( Ric_\kappa{}^\eta (K_{\mu\eta\lambda} - K_{\mu\lambda\eta}) + \frac{1}{2} g_{\lambda\mu} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} (K^\xi{}_{\eta\zeta} - K^\xi{}_{\zeta\eta}) + \frac{1}{2} g_{\mu\lambda} Ric_\xi{}^\eta K^\xi{}_{\eta\kappa} \\
& + g_{\mu\lambda} Ric_\kappa{}^\eta K^\xi{}_{\xi\eta} - K^\xi{}_{\xi\lambda} Ric_{\kappa\mu} + \frac{1}{2} g_{\mu\lambda} Ric_\kappa{}^\xi (K^\eta{}_{\xi\eta} - K^\eta{}_{\eta\xi}) \Big) \\
- \quad & d_7 \Big( Ric_\lambda{}^\eta (K_{\mu\eta\kappa} - K_{\mu\kappa\eta}) + \frac{1}{2} g_{\kappa\mu} \mathcal{W}^{\kappa\zeta}{}_{\lambda\xi} (K^\xi{}_{\eta\zeta} - K^\xi{}_{\zeta\eta}) + \frac{1}{2} g_{\mu\kappa} Ric_\xi{}^\eta K^\xi{}_{\eta\lambda} \\
& + g_{\kappa\mu} Ric_\lambda{}^\eta K^\xi{}_{\xi\eta} - K^\xi{}_{\xi\kappa} Ric_{\lambda\mu} + \frac{1}{2} g_{\mu\kappa} Ric_\lambda{}^\xi (K^\eta{}_{\xi\eta} - K^\eta{}_{\eta\xi}) \Big) \\
+ \quad & b_{10} \Big( g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} (K^\xi{}_{\zeta\eta} - K^\xi{}_{\eta\zeta}) + g_{\mu\kappa} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} (K^\xi{}_{\eta\zeta} - K^\xi{}_{\zeta\eta}) \\
& + g_{\mu\lambda} Ric_\kappa{}^\xi (K^\eta{}_{\eta\xi} - K^\eta{}_{\xi\eta}) + g_{\mu\kappa} Ric_\lambda{}^\xi (K^\eta{}_{\xi\eta} - K^\eta{}_{\eta\xi}) \\
& + g_{\kappa\mu} Ric_\lambda{}^\eta K^\xi{}_{\xi\eta} - g_{\lambda\mu} Ric_\kappa{}^\eta K^\xi{}_{\xi\eta} + Ric_{\mu\kappa} K^\eta{}_{\lambda\eta} - Ric_{\mu\lambda} K^\eta{}_{\kappa\eta} \Big) \\
+ \quad & 2 b_{10} \Big( \mathcal{W}^\eta{}_{\mu\kappa\xi} (K^\xi{}_{\eta\lambda} - K^\xi{}_{\lambda\eta}) + \mathcal{W}^\eta{}_{\mu\lambda\xi} (K^\xi{}_{\kappa\eta} - K^\xi{}_{\eta\kappa}) \\
& - K_{\mu\xi\eta} \mathcal{W}^{\eta\xi}{}_{\kappa\lambda} - K^\xi{}_{\xi\eta} \mathcal{W}^\eta{}_{\mu\lambda\kappa} \Big) \qquad (5)
\end{aligned}
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where $d_1, d_3, d_6, d_7, b_{10}$ are some real constants.

# The first task

We are going to try to generalize pp–waves as follows

**Conjecture**

*There exists a new class of spacetimes with pp–metric and purely axial torsion which are solutions of the field equations (2), (3).*

Expectations:

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# Physical interpretation

Massless Dirac action:

$$S_{neutrino} := 2i \int \left( \xi^a \sigma^\mu{}_{a\dot{b}} (\nabla_\mu \bar{\xi}^{\dot{b}}) - (\nabla_\mu \xi^a) \sigma^\mu{}_{a\dot{b}} \bar{\xi}^{\dot{b}} \right).$$

In Einstein–Weyl theory the action is given by:

$$S_{EW} = S_{neutrino} + k \int \mathcal{R}.$$

We obtain the well known Einstein–Weyl field equations

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# The massless Dirac operator

The massless Dirac operator is the matrix operator

$$W = -i\sigma^\alpha \left( \frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left( \frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ {\beta \atop \alpha\gamma} \right\} \sigma^\gamma \right) \right). \tag{8}$$

The massless Dirac operator (8) describes a single massless neutrino living in 3-dimensional compact universe $M$.

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Let $M$ be a 3-dimensional connected oriented manifold equipped with a Riemnnian metric $g_{\alpha\beta}$ and let $W$ be the corresponding massless Dirac operator (8).

Two basic examples when the spectrum of $W$ can be calculated explicitly:

- the unit torus $\mathbb{T}^3$ equipped with Euclidean metric.
- the unit sphere $\mathbb{S}^3$ equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in Euclidean space $\mathbb{R}^4$.

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For a general oriented Riemannian 3-manifold there is no reason for the spectrum of massless Dirac operator $W$ to be symmetric (M. F. Atiyah, V. K. Patodi and I. M. Singer).

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Pfäffle: the example based on the idea of choosing a 3-manifold with flat metric but highly nontrivial toplogy!

*Barakovic*, *Pasic*, *Vassiliev*: the simplest possible topology and construct an explicit example of spectral asymmetry by perturbing the metric!

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