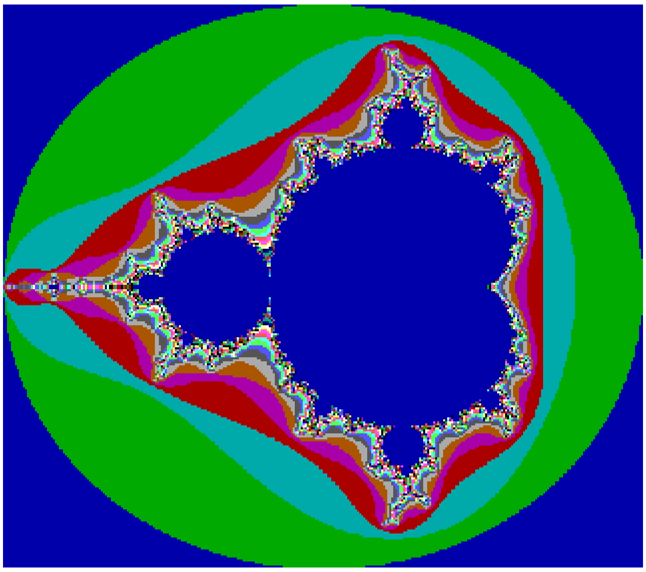
Mathematics Seminar Project

<u>FRACTAL DIMENSION</u>



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1. Introduction

In the past, mathematics has been largely concerned with sets and functions to which the methods of classical calculus can be applied. Sets of functions that are not sufficiently smooth or regular tended to be ignored. In recent years this attitude has changed. It has been realised that a great deal can be said about the mathematics of non-smooth sets. Fractal geometry provides a general framework for the study of such irregular sets.

Although people hear about *"fractals"* all the time, still most do not understand what they are and what they represent. Many attempts have been made to define fractals in a purely mathematical sense, but such definitions have often proved to be unsatisfactory in a general context. Still, fractal geometry provides a number of techniques for dealing with fractals, and this essay covers only a small part of it, i.e. it deals only with the notion of fractal dimension.

As I already mentioned, methods of classical geometry and calculus are unsuited to studying fractals. The main tool of fractal geometry is dimension in its many forms. We are all familiar with the idea that a curve is a 1 – dimensional object, and a surface is 2 – dimensional. Let us now consider the following set:

Let E_0 be a line segment of unit length. The set E_1 consists of the four segments obtained by removing the middle third of E_0 and replacing it by the other two sides of the equilateral triangle based on the removed segment. We construct E_2 by repeating the same procedure on each of the segments in E_1, and so on. Thus E_k comes from replacing the middle third of each straight line segment of E_{k-1} by the other two sides of the equilateral triangle. As k tends to infinity, the sequence of polygonal curves E_k approaches a limiting curve F, called *the Von Koch curve* (Figure 1.1).

The following argument gives a rather crude interpretation of what the dimension of this set is, indicating how it reflects scaling properties and self – similarity. A Figure 1.1 indicates, the Von Koch curve is made up of four copies of itself scaled by a factor 1/3, and has dimension $d = -ln\ 4/ln\ (1/3) = ln\ 4/ln\ 3 = 1.262$. In general, a set made up of m copies of itself scaled by a factor r might be thought of as having dimension $d = -\ln m/\ln r$. The number obtained in this way is usually referred to as the *similarity dimension* of the set [1].

Unfortunately, similarity dimension only makes sense for a small number of self similar sets, and cannot be applied to a vast number of sets which are considered to be very important. Nevertheless, there are other definitions of dimension that are much more widely applicable. For example, Hausdorff dimension and the Minkowski dimension may be defined for any sets, and it may be shown that they equal the similarity dimension (see latter sections).

[1] *If you would like to visualise the forming of the von Koch curve and some other fractals, look* http://www.wmin.ac.uk/~storyh/fractal/frac.html

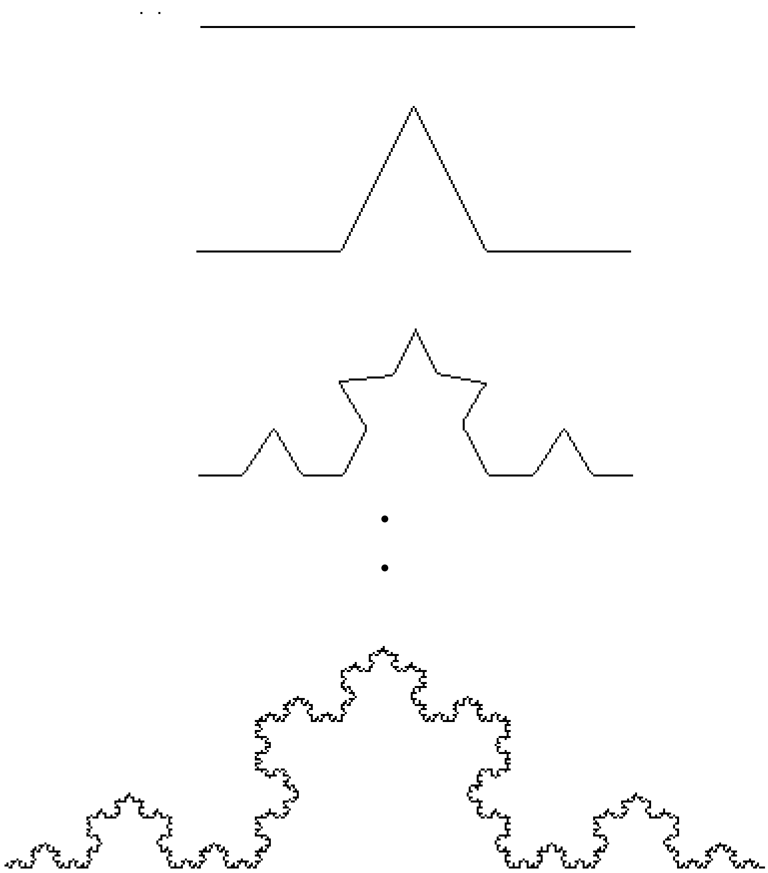

FIGURE 1.1 Construction of the Von Koch curve F. $d(F) = \ln 4 / \ln 3 = 1.262$ [2]

As mentioned above, it is not very easy to rigorously define a *fractal*. The name itself was given to highly irregular sets by Benoit Mandelbrot in his fundamental essay in 1975. Mandelbrot defined a fractal to be a set with Hausdorff dimension strictly greater than its topological dimension (the *topological* dimension of a set is an integer). Although this essay covers just a small part of fractal geometry, it would be useful if we define what we mean when we refer to a set F as a fractal. Falconer[3] states the following non - rigorous definition : We consider a set *F* in Euclidean space to be fractal if it has all or most of the following properties:

(i) F has a fine structure, i.e. detail on arbitrarily small scales
(ii) F is too irregular to be described in traditional geometrical language, both locally and globally.
(iii) Often F has some form of self – similarity, perhaps approximate or statistical.
(iv) Usually, the "fractal dimension" of F (defined in some way) is greater than its topological dimension.
(v) In many cases of interest F has a very simple, perhaps recursive definition.
(vi) Often F has a natural appearance.

Now I embark on describing in more detail what the *fractal dimension* actually is.

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[2] *Look http://hyperion.advanced.org/3493/noframes/fractal.html for more simple examples & picutures*
[3] *In the book "Techniques in Fractal Geometry", page XI*

2. Hausdorff measure and dimension

Felix Hausdorff (1869 – 1942) [4]

Of the wide variety of 'fractal dimensions', the Hausdorff definition is probably the oldest, and it has the advantage of being defined for any set. It is also mathematically convenient, because it uses the notion of measures, which are relatively easy to manipulate. For the understanding of fractal dimension and fractal geometry in general, the understanding of Hausdorff dimension is very important.

Hausdorff measure

Recall that if U is any non-empty subset of n – dimensional Euclidean space, $\mathbb{R}^n$, the *diameter* of U is defined as $|U| = \sup\{ \mid x - y \mid : x, y \in U\}$. If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most ε that cover F, i.e. $F \subset \cup^{\infty}_{i=1} U_i$ with $0 < |U_i| \le \varepsilon$ for each i, we say that $\{U_i\}$ is a ε - *cover* of F.

Suppose that F is a subset of $\mathbb{R}^n$ and s is a non – negative number. For any $\varepsilon>0$ we define

$$\boldsymbol{H}^s_\varepsilon(F) = \inf \{\textstyle\sum_i |U_i|^s : \{U_i\} \text{ is a } \varepsilon\text{ - cover of } F\}.$$

We write

$$H^s(F) = \lim_{\varepsilon \to 0} H^s_\varepsilon(F)$$

We call $\boldsymbol{H}^s(F)$ the *s-dimensional Hausdorff measure of F.*

Hausdorff measure generalises the ideas of length, area, volume, etc. It may be shown that, for subsets of $\mathbb{R}^n$, n-dimensional Hausdorff measure is, to within a constant multiple, just n-dimensional Lebesgue measure, i.e. the n-dimensional volume. So, $\boldsymbol{H}^n(F) = c_n \text{ vol}^n (F)$, where the constant $c_n = \pi^{1/2\ n} / 2^n (½ n)!$ is the volume of an n-dimensional ball of diameter 1. So we have that $\boldsymbol{H}^0(F)$ is the number of points in F; $\boldsymbol{H}^1(F)$ gives the length of a smooth curve F; $\boldsymbol{H}^2(F)= ¼\, \pi$ x area(F) if F is a smooth surface and $\boldsymbol{H}^m(F)=c_m$ x $\text{vol}^m(F)$ if F is a smooth m-dimensional submanifold of $\mathbb{R}^n$ (i.e. m-dimensional surface in the classical sense).

[4] *For his biography and research, look* http://history.math.csusb.edu/Mathematicians/Hausdorff.html

The following is called the scaling property of the Hausdorff measure:

If $F \subset \mathbb{R}^n$ *and* $\lambda > 0$ *then*

$$\boldsymbol{H}^s(\lambda F) = \lambda^s \, \boldsymbol{H}^s(F)$$

where $\lambda F = \{\lambda x : x \in F\}$*, i.e. the set F scaled by a factor* λ [5].

<u>*Hausdorff dimension*</u>

The Hausdorff dimension (sometimes referred to as *Hausdorff – Besicovich dimension*) is defined formally in the following way:

$$\dim_H F = \inf\{s: \boldsymbol{H}^s(F) = 0\} = \sup\{s: \boldsymbol{H}^s(F) = \infty\}$$

so that $\boldsymbol{H}^s(F) = \infty$ if $s < \dim_H F$ and $\boldsymbol{H}^s(F) = 0$ if $s > \dim_H F$.

What this means is that there is a critical value of s at which $\boldsymbol{H}^s(F)$ 'jumps' from ∞ to 0, and this value of s is the *Hausdorff dimension* of F.

There are a few properties of this dimension which are worth mentioning[6]:

(i) If $F \subset \mathbb{R}^n$ is open, then $\dim_H F = n$, since F contains a ball of positive n-dimensional volume;
(ii) If F is continuously differentiable m – dimensional submanifold (i.e. m-dimensional surface) of $\mathbb{R}^n$ then $\dim_H F = m$;
(iii) If $E \subset F$ then $\dim_H E \leq \dim_H F$;
(iv) If $F_1, F_2, \ldots$ is a (countable) sequence of sets, then $\dim_H \cup_{i=1}^{\infty} F_i = \sup_{1 \leq i < \infty} \{\dim_H F_i\}$;
(v) If F is countable then $\dim_H F = 0$.

Here follow a few simple examples to illustrate how the Hausdorff dimension can be calculated.

[5] *Proof of this fact can be found in "Fractal Geometry", page 27*
[6] *According to Falconer, "Fractal geometry", page 29.*

Example 2.1

Let $F = \{ \underline{0} \} \subset \mathbb{R}^n$. It is obvious that $\boldsymbol{H}^0(F)=1$ and $\boldsymbol{H}^1(F)=0$, so $\dim_H F = 0$. Of course, this was obvious from the beginning, since F is countable, and by property (v), all countable sets F have $\dim_H F = 0$.

Example 2.2

Let $F = [0, 1] \subset \mathbb{R}$. From familiar properties of length and area, $\boldsymbol{H}^0(F)$ = number of points in $F = \infty$, $0 < \boldsymbol{H}^1(F)$ = length(F) = 1 and $\boldsymbol{H}^2(F)$ = area(F) $\times \frac{1}{4}\pi = 0$. So we have

$$\dim_H F = \inf\{s: \boldsymbol{H}^s(F) = 0\} = \sup\{s: \boldsymbol{H}^s(F) = \infty\} = 1,$$

with $\boldsymbol{H}^s(F) = \infty$ for $s < 1$ and $\boldsymbol{H}^s(F) = 0$ for $s > 1$. This result is also hardly surprising, since by the property (ii) of the Haudorff dimension, the Hausdorff dimension of F is in this case 1.

Example 2.3

Let $F = \{ \underline{x} : x_1 \in [0, 1], x_2 = 0\} \subset \mathbb{R}^2$. Since this is obviously just the line on the x_1 axis of length 1, $\boldsymbol{H}^0(F)$ = number of points in $F = \infty$, $0 < \boldsymbol{H}^1(F)$ = length(F) = 1 and $\boldsymbol{H}^2(F)$ = area(F) $\times \frac{1}{4}\pi = 0$, so the $\dim_H F = 1$. Of course, this too was obvious from the start, since again by the property (ii) of the Hausdorff dimension, F has Hausdorff dimension 1.

Example 2.4

This is the only not completely trivial example that we are going to consider in this section[7].

Let F be the middle third Cantor set (see Figure 2.1).If s=ln 2 / ln 3 = 0.6309... then $\dim_H F = s$ *and* $\frac{1}{2} \le \boldsymbol{H}^s(F) \le 1$.

[7] *Taken from Falconers "Fractal Geometry", page 31.*

FIGURE 2.1 Construction of the middle third Cantor set F .

Heuristic calculation. The Cantor set F splits into a left part $F_L = F \cap [0, 1/3]$ and a right part $F_R = F \cap [2/3, 1]$. Clearly both parts are geometrically similar to F but scaled by a ratio 1/3, and $F = F_L \cup F_R$ with this union disjoint. Thus for any s

$$\boldsymbol{H}^s(F) = \boldsymbol{H}^s(F_L) + \boldsymbol{H}^s(F_R) = (1/3)^s\, \boldsymbol{H}^s(F) + (1/3)^s\, \boldsymbol{H}^s(F)$$

by the scaling property of Haudorff measures. *Assuming that at the critical value* $s = \dim_H F$ *we have* $0 < \boldsymbol{H}^s(F) < \infty$ (a big assumption, but one that can be justified) we may divide by $\boldsymbol{H}^s(F)$ to get $1 = 2\,(1/3)^s$ or $s = \ln 2 / \ln 3$.

Rigorous calculation. ....[8]

[8] *For a rigorous proof of this fact, look up "Fractal Geometry", page 32.*

3. Minkowski dimension

(also called box-counting or Bouligand-Minkowski dimension)

Hermann Minkowski (1864 – 1909) [9]

In the last part we have seen that the calculation of Hausdorff measures can be a little involved, even for simple sets. So we are interested in finding a different definition of dimension which would be more applicable in calculating the dimension of a set F. But there are no hard and fast rules for deciding whether a quantity may reasonably be regarded as a dimension. The factors that determine the acceptability of a definition of a dimension are recognised largely by experience and intuition.

It should not be assumed that different definitions of dimension give the same value of a dimension for all sets, even for those that are considered 'nice'. So, the notion of a 'dimension' of a set should be separated from the notion of a 'definition of a dimension'. In this part of the essay, we shall introduce another definition of a dimension, the *Minkowski* or the *box-counting dimension* (for simplicity, I shall only use the term 'Minkowski dimension' from now on).

The Minkowski dimension is one of the most widely used dimensions. It is reasonably easy to calculate, and the notion of 'measures' is avoided. There are several versions of this definition, and the example calculations will be based on just one of them. However, it would be useful if some other versions are mentioned, so here follows the first one[10]:

Let F be any non-empty bounded set of $\mathbb{R}^n$ *. The lower and upper Minkowski dimensions of F are given by*

$$\underline{\dim}_M F = \underline{\lim}_{\varepsilon \to 0} \frac{\ln N_\varepsilon(F)}{-\ln \varepsilon}$$

$$\overline{\dim}_M F = \overline{\lim}_{\varepsilon \to 0} \frac{\ln N_\varepsilon(F)}{-\ln \varepsilon}$$

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[9] *For his biography and reasearch, look* http://history.math.csusb.edu/Mathematicians/Minkowski.html

and the Minkowski dimension of F by

$$\dim_M F = \lim_{\varepsilon \to 0} \frac{\ln N_\varepsilon(F)}{-\ln \varepsilon}$$

(if this limit exists), where $N_\varepsilon(F)$ is any of the following:

(i) *the smallest number of closed balls of radius ε that cover F;*
(ii) *the smallest number of cubes of side ε that cover F (box-counting);*
(iii) *the number of ε-mesh cubes that intersect F;*
(iv) *the smallest number of sets of diameter at most ε that cover F;*
(v) *the largest number of disjoint balls of radius ε with centres in F.*

This is a very useful definition, but not very 'friendly' when it comes to calculating the actual Minkowski dimension. However, there is an equivalent definition of the Minkowski dimension that is of rather different form. Before we actually give it, let us firstly recall that the *ε - neighbourhood (or the ε - parallel body) F_ε* of *F* is

$$F_\varepsilon = \{ x \in \mathbb{R}^n : dist\,(x, F) < \varepsilon \},$$

where *dist (x, F)* is the Euclidean distance in $\mathbb{R}^n$, i.e.

$$dist(x, F) := \inf_{y \in F} dist(x, y)$$

So the ε-neighbourhood F_ε is the set of points within distance ε of *F*. It sometimes referred to as the *Minkowski sausage.*

So now, we can formulate the following proposition:

If F is a subset of $\mathbb{R}^n$, *then*

$$\underline{\dim}_M F = n - \overline{\lim_{\varepsilon \to 0}} \frac{\ln vol^n(F_\varepsilon)}{\ln \varepsilon}$$

$$\overline{\dim}_M F = n - \underline{\lim}_{\varepsilon \to 0} \frac{\ln vol^n(F_\varepsilon)}{\ln \varepsilon}$$

where F_ε is the ε-neighbourhood of F, and $vol^n(F_\varepsilon)$ is its n-dimensional volume.

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[10] *According to Falconer, "Fractal Geometry", page 38.*

Proof: ....[11]

There is a very important relation between the Minkowski and Hausdorff dimension. If F can be covered by $N_\varepsilon(F)$ sets of diameter ε, then, from the scaling property of the Hausdorff measure,

$$\boldsymbol{H}^s_\varepsilon(F) \leq N_\varepsilon(F)\ \varepsilon^s.$$

If $1 < \boldsymbol{H}^s(F) = \lim_{\varepsilon\to 0} \boldsymbol{H}^s_\varepsilon(F)$ then $\ln N_\varepsilon(F) + s \ln \varepsilon > 0$ if ε is sufficiently small. Thus $s \leq \underline{\lim}_{\varepsilon\to 0} \ln N_\varepsilon(F) / -\ln \varepsilon$ so

$$\dim_H F \leq \underline{\dim}_M F \leq \overline{\dim}_M F$$

for any $F \subset \mathbb{R}^n$. We do not generally get equality here, and there are plenty of examples where this inequality is strict.

Now let us look at a couple of examples [12] illustrating how the Minkowski dimension is calculated. Note that in general, the second version of the Minkowski dimension will be used.

Example 3.1 Let $F = \{ \underline{x} \in \mathbb{R}^2 : |\underline{x}| = 1\} \subset \mathbb{R}^2$. Let us first calculate the ε - neighbourhood of F. Since F is obviously a circle of radius 1, centred at $\underline{0}$, the ε - neighbourhood of F is the 'ring' around the circle F, where all the points in F_ε satisfy

$$F_\varepsilon = \{ \underline{y} \in \mathbb{R}^2 : \mathrm{dist}(\underline{x},\underline{y}) < \varepsilon, \text{ for } \underline{x} \in F\}.$$

The area of this F_ε is equal to the difference between the areas of the circles with radii 1+ ε and 1 - ε, centred at the origin $\underline{0}$, and this area is the 2-dimensional volume of the ε - neighbourhood, $\mathrm{vol}^2(F_\varepsilon)$.

$$\mathrm{vol}^2(F_\varepsilon) = \pi(1+\varepsilon)^2 + \pi(1-\varepsilon)^2 = 4\pi\varepsilon$$

So the Minkowski dimension of F is then

$$\dim_M F = 2 - \lim_{\varepsilon\to 0} \frac{\ln vol^2(F_\varepsilon)}{\ln \varepsilon} = 2 - \lim_{\varepsilon\to 0} \frac{\ln 4\pi\varepsilon}{\ln\varepsilon} = 2 - \lim_{\varepsilon\to 0} \frac{\ln 4\pi + \ln\varepsilon}{\ln\varepsilon} =$$

$$= 2 - (0+1) = 1$$

So this F has dimension 1, which was expected, since it is a curve.

[11] *The proof can be found in "Fractal Geometry", page 42.*

[12] *Thanks to Professor D.G.Vassiliev for his invaluable guidance and help in this section.*

Example 3.2 Let F = { $\underline{x} \in \mathbb{R}^2$: | $\underline{x}$ | ≤ 1} ⊂ $\mathbb{R}^2$. F is a disk of radius 1 centred at the origin. Its ε - neighbourhood is also a disk but with radius 1+ε centred at the origin. The 2-dimensional volume of F is therefore

$$vol^2 (F_\varepsilon) = \pi (1 + \varepsilon)^2$$

The Minkowski dimension of F is therefore

$$\dim_M F = 2 - \lim_{\varepsilon \to 0} \frac{\ln vol^2 (F_\varepsilon)}{\ln \varepsilon} = 2 - \lim_{\varepsilon \to 0} \frac{\ln \pi (1+\varepsilon)^2}{\ln \varepsilon} = 2 - \lim_{\varepsilon \to 0} \left(\frac{\ln \pi}{\ln \varepsilon} + \frac{2 \ln(1+\varepsilon)}{\ln \varepsilon} \right) =$$

$$= 2 - (0+0) = 2$$

So the dimension of F is 2, which was expected, since F is a disk.

Example 3.3 [13] Let F = { 1/n } ⊂ $\mathbb{R}$ (n=1, 2, 3, ...). We get the ε - neighbourhood of F in this case by putting a small interval [1/k - ε, 1/k + ε] around every element 1/k of F. After a while, for n = N ∈ $\mathbb{N}$, the intervals [1/(N+1) - ε, 1/(N+1) + ε]
and [1/N - ε, 1/N + ε] will start to overlap, so we will get one interval [-ε, 1/N + ε], while for n<N, the intervals remain disjoint and with length 2ε. So the 1-dimensional volume (length) of the ε - neighbourhood of F is

$$vol^1(F_\varepsilon) = \sum_{i=1}^{N-1} 2\varepsilon + length\left[-\varepsilon, \frac{1}{N} + \varepsilon \right] = (N-1)2\varepsilon + \frac{1}{N} + 2\varepsilon = 2\varepsilon N + \frac{1}{N}$$

Now, we must find an N, for which

$$1/(N+1) + \varepsilon \leq 1/N - \varepsilon \Leftrightarrow 1/(N+1) - 1/N \leq -2\varepsilon \Leftrightarrow 1/N(N+1) \geq 2\varepsilon \Leftrightarrow N^2 + N - 1/(2\varepsilon) \leq 0$$

A simple calculation shows that minimum N that satisfies this equation equals the integer part of

$$-\frac{1}{2} + \sqrt{\frac{\varepsilon + 2}{4\varepsilon}}$$

[13] *Look up "Fractal Geometry", page 45 for a different solution of this example*

Now we need to find inequalities for N and 1/N. Since N is the integer part of the above number, we know that

$$N \le -\frac{1}{2} + \sqrt{\frac{\varepsilon+2}{4\varepsilon}} \qquad\qquad N \ge -\frac{1}{2} + \sqrt{\frac{\varepsilon+2}{4\varepsilon}} - 1 = -\frac{3}{2} + \sqrt{\frac{\varepsilon+2}{4\varepsilon}}$$

and a simple transformation of these two inequalities yields

$$\frac{1}{N} \ge \varepsilon + \sqrt{\varepsilon^2 + \varepsilon} \qquad\qquad \frac{1}{N} \le \frac{3\varepsilon + \sqrt{\varepsilon^2 + 2\varepsilon}}{-4\varepsilon + 1}$$

Putting these inequalities into the equation for the 1-dimensional volume of the ε - neighbourhood of F, very easily we get the following inequalities

$$2\sqrt{\varepsilon^2 + 2\varepsilon} - 2\varepsilon \le vol^1(F_\varepsilon) = 2\varepsilon N + \frac{1}{N} \le -\varepsilon + \sqrt{\varepsilon^2 + 2\varepsilon} + \frac{3\varepsilon + \sqrt{\varepsilon^2 + 2\varepsilon}}{-4\varepsilon + 1}$$

or

$$2\sqrt{2\varepsilon}\left(\sqrt{1+\frac{\varepsilon}{2}} - \sqrt{\frac{\varepsilon}{2}}\right) \le vol^1(F_\varepsilon) \le \sqrt{2\varepsilon}\left(-\sqrt{\frac{\varepsilon}{2}} + \sqrt{\frac{\varepsilon}{2}+1} + \frac{\sqrt{\frac{3\varepsilon}{2}} + \sqrt{\frac{\varepsilon}{2}+1}}{-4\varepsilon+1}\right)$$

If we take logarithms of these terms, divide through with *ln ε* and take the limit as ε→0 of the right hand side inequality, and the left hand side inequality, we get

$$\lim_{\varepsilon\to 0}\frac{\ln vol^1(F_\varepsilon)}{\ln\varepsilon} \le \lim_{\varepsilon\to 0}\frac{\ln\left(2\sqrt{2\varepsilon}\right)}{\ln\varepsilon} + \lim_{\varepsilon\to 0}\frac{\ln\left(\sqrt{1+\frac{\varepsilon}{2}} - \sqrt{\frac{\varepsilon}{2}}\right)}{\ln\varepsilon} = \lim_{\varepsilon\to 0}\left(\frac{\ln 2\sqrt{2}}{\ln\varepsilon} - \frac{\frac{1}{2}\ln\varepsilon}{\ln\varepsilon}\right) + 0 = 0 + \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{\varepsilon\to 0}\frac{\ln vol^1(F_\varepsilon)}{\ln\varepsilon} \ge \lim_{\varepsilon\to 0}\left(\frac{\ln\sqrt{2}}{\ln\varepsilon} + \frac{\frac{1}{2}\ln\varepsilon}{\ln\varepsilon} + \frac{\ln\left(-\sqrt{\varepsilon/2} + \sqrt{\varepsilon/2+1} + \frac{\sqrt{3\varepsilon/2} + \sqrt{\varepsilon/2+1}}{-4\varepsilon+1}\right)}{\ln\varepsilon}\right) = 0 + \frac{1}{2} + 0 = \frac{1}{2}$$

So, finally, we get the Minkowski dimension of this set:

$$\underline{\dim_M F = 1 - ½ = ½}$$

No one would regard this set, with all of its points isolated, as a fractal, and yet it has fractional Minkowski dimension of ½ .

Example 3.4 The following example is rather harder to do rigorously, so some approximations have to be made. Let $F = \{ 1 / n^{\alpha}\} \subset \mathbb{R}$, where $\alpha > 0$. We get the ε - neighbourhood of F in this case by putting a small interval $[1/k^{\alpha} - \varepsilon, 1/k^{\alpha} + \varepsilon]$ around every element $1/k^{\alpha}$ of F. After a while, for $n=N \in \mathbb{N}$, the intervals $[1/(N+1)^{\alpha} - \varepsilon, 1/(N+1)^{\alpha} + \varepsilon]$ and $[1/N^{\alpha} - \varepsilon, 1/N^{\alpha} + \varepsilon]$ will start to overlap, so we will get one interval $[-\varepsilon, 1/N^{\alpha} + \varepsilon]$, while for n<N, the intervals remain disjoint and with length 2ε. So the 1-dimensional volume (length) of the ε - neighbourhood of F is

$$vol^1(F_\varepsilon) = \sum_{i=1}^{N-1} 2\varepsilon + length\left[-\varepsilon, \frac{1}{N^\alpha} + \varepsilon\right] = (N-1)2\varepsilon + \frac{1}{N^\alpha} + 2\varepsilon = 2\varepsilon N + \frac{1}{N^\alpha}$$

Now, we must find an N, for which

$$\frac{1}{N^\alpha} - \frac{1}{(N+1)^\alpha} \approx 2\varepsilon$$

Since $1 / (N + 1)^{\alpha}$ can be approximated by

$$\frac{1}{(N+1)^\alpha} = \frac{1}{N^\alpha}\frac{1}{\left(1+\frac{1}{N}\right)^\alpha} = \frac{1}{N^\alpha}\left(1+\frac{1}{N}\right)^{-\alpha} \approx \frac{1}{N^\alpha}\left(1-\frac{\alpha}{N}\right)$$

we get that

$$\frac{1}{N^\alpha} - \frac{1}{(N+1)^\alpha} \approx \frac{1}{N^\alpha} - \frac{1}{N^\alpha}\left(1-\frac{\alpha}{N}\right) = \frac{\alpha}{N^{\alpha+1}} \approx 2\varepsilon$$

which can be easily solved to find the N which approximately satisfies the equation above, to get

$$N \approx \left(\frac{\alpha}{2\varepsilon}\right)^{\frac{1}{\alpha+1}}$$

Putting this into the equation for the 1 - dimensional volume of the ε - neighbourhood F_ε, we get that

$$vol^1(F_\varepsilon) \approx 2\varepsilon\left(\frac{\alpha}{2\varepsilon}\right)^{\frac{1}{\alpha+1}} + \left(\frac{2\varepsilon}{\alpha}\right)^{\frac{\alpha}{\alpha+1}} = (2\varepsilon)^{\frac{\alpha}{\alpha+1}}\left[(\alpha)^{\frac{1}{\alpha+1}} + \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha+1}}\right]$$

Finally, we can find the Minkowski dimension of F in the following way

$$\dim_M F = 1 - \lim_{\varepsilon\to 0}\frac{\ln vol^1 F_\varepsilon}{\ln\varepsilon} = 1 - \left[\lim_{\varepsilon\to 0}\left(\frac{\frac{\alpha}{\alpha+1}\ln\varepsilon}{\ln\varepsilon} + \frac{\ln\left[\alpha^{\frac{1}{\alpha+1}} + \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha+1}}\right]}{\ln\varepsilon}\right)\right] = 1 - \left(\frac{\alpha}{\alpha+1} + 0\right) = \frac{1}{\alpha+1}$$

So, the dimension of F is 1/(α+1), for α > 0, which corresponds to the previous example, where the case was α=1. Again we get that the dimension of the set is fractal, and yet this set at the first look would not appear to be a fractal.

With this I close the discussion about the (ordinary) Minkowski dimension, noting once again that it is in some cases much easier to calculate than the Hausdorff dimension, although there are cases where the Hausdorff dimension is the right choice. Minkowski dimension, however, is used to define the ‘interior dimension’ of the boundary of *F*, and that is what the next and final part of this essay covers.

4. Interior Minkowski dimension of the boundary

The notion of the *'interior Minkowski dimension'* is a very important notion in fractal geometry, since it is connected with the problem of the *eigenvalue counting function*,

$$N(\lambda) = \# \{ \lambda_j (\Omega) < \lambda \},$$

where Ω is a bounded open set in $\mathbb{R}^n$ ($n \geq 2$), with fractal boundary $\partial\Omega$ [14]. Here I don't go deeper into the this particular problem, but the notion of the interior Minkowski dimension should be explained further.

First we need to define the *interior* ε *- neighbourhood of* the boundary $\partial\Omega$ of a bounded open set Ω in $\mathbb{R}^n$ ($n \geq 2$) is defined by

$$\partial\Omega_\varepsilon^i = \{ \underline{x} \in \Omega : \text{dist} (\underline{x}, \partial\Omega) < \varepsilon\}$$

where dist(. , .) denotes the Euclidean distance in $\mathbb{R}^n$.

Having this in mind, let us state the definition of the *interior Minkowski dimension*:

Let Ω be a bounded open set in $\mathbb{R}^n$ ($n \geq 2$), with very irregular (fractal) boundary $\partial\Omega$. Then the interior Minkowski dimension of $\partial\Omega$ is

$$\dim_I \partial\Omega = n - \lim_{\varepsilon \to 0} \frac{\ln vol^n \partial\Omega_\varepsilon^i}{\ln \varepsilon}$$

where $\partial\Omega_\varepsilon^i$ is the interior ε - neighbourhood of the boundary $\partial\Omega$.

I will not go much deeper into this problem, but here follow a couple of examples (one simple and one more interesting) which use this notion of the interior dimension of the boundary. The first one we have already seen in the previous section (Example 3.3), but this time the example will be modified to fit this notion.

[14] *For further details on this, look Fleckinger – Pellé and Vassiliev, "An example...", or "Techniques in Fractal Geometry", page 226.*

Example 4.1 Let $\Omega = \{ \underline{x} \in \mathbb{R}^2 : | \underline{x} | \leq 1\} \subset \mathbb{R}^2$. Let us find the interior Minkowski dimension of the boundary of Ω. First, we need to define the boundary $\partial\Omega$. We get that

$$\partial\Omega = \{ \underline{x} \in \mathbb{R}^2 : | \underline{x} | = 1\} \subset \mathbb{R}^2$$

This is a circle $\mathbb{R}^2$ with radius 1 and centered at $\underline{0}$. And so the interior ε - neighbourhood of $\partial\Omega$ is

$$\partial\Omega_\varepsilon^{\mathrm{i}} = \{ \underline{x} \in \Omega : \mathrm{dist}(\underline{x}, \partial\Omega) < \varepsilon\}.$$

This is a 'ring' that lies between two circles centered at $\underline{0}$, with radii 1 and (1 - ε). So the 2 – dimensional volume (area) of the ε - neighbourhood $\partial\Omega_\varepsilon^{\mathrm{i}}$ is

$$vol^2 \partial\Omega_\varepsilon^i = \pi - \pi(1-\varepsilon)^2 = \pi\varepsilon(2-\varepsilon)$$

So finally we get the interior Minkowski dimension of $\partial\Omega$

$$\dim_I \partial\Omega = 2 - \lim_{\varepsilon\to 0} \frac{\ln vol^2 \partial\Omega_\varepsilon^i}{\ln\varepsilon} = 2 - \lim_{\varepsilon\to 0}\left(\frac{\ln\varepsilon}{\ln\varepsilon} + \frac{\ln\pi}{\ln\varepsilon} + \frac{\ln(2-\varepsilon)}{\ln\varepsilon} \right) = 2 - (1+0+0) = 1$$

So the interior Minkowski dimension of $\partial\Omega$ is 1, which was only to be expected, since it is a curve.

Example 4.2 [15] Let s be a positive given number satisfying

$$1 + \sqrt{2} < s < 3$$

Let us consider in $\mathbb{R}^2$ the open set Q, which consists of a union of open squares. The central square Q_0 has side 1. The side of each of the 4 consecutive squares Q_1 is s times smaller; these squares are "sticked" on the middles of the sides of Q_0.

We now have 4 × 3 "free" sides with length s^{-1}; on each middle part of the sides we "stick" again one square Q_2 with side s^{-2} etc. At the kth step we have

$$n_k = 4/3 \times 3^k, \qquad k \geq 1, \text{ and } n_0 = 1$$

squares Q_k with sides s^{-k}.

[15] *Taken from Fleckinger – Pellé and Vassiliev, "An example..."*

We denote by Q the union of all these squares for $k = 0, 1, 2, ...$. Note that Q is disconnected; moreover it follows from the fact $1 + \sqrt{2} < s < 3$ that the squares do not overlap and that Q is with finite measure.

The interior Minkowski dimension d_i of ∂Q is

$$d_i = \ln 3 / \ln s.$$

This can be derived by a simple calculation, since for a given $\varepsilon > 0$

$$vol^2\left(\partial\Omega_\varepsilon^i\right) = \sum_{k=0}^{K} n_k \left(4\varepsilon s^{-k} - 4\varepsilon^2\right) + \sum_{k=K+1}^{+\infty} n_k s^{-2k}$$

where k is such that

$$s^{-(K+1)} < 2\,\varepsilon \leq s^{-K}.$$

Note that it follows from the first equation ($1 + \sqrt{2} < s < 3$) that $1 < d_i < 2$.

With that, I finish off this section about the interior Minkowski dimension of the boundary. Considering that this area of fractal geometry was not covered in much depth, I suggest that you look up one of the works mentioned in the Bibliography for further reference.

5. Conclusion

Although this essay does not require a conclusion as such, I think it would be only appropriate to sum up the impressions I got while working on it.

Fractal geometry is an extraordinary part of mathematics, and its mathematical background is not as recent as people would expect. On the other hand, the potential of the use of this branch of mathematics has only been discovered some years ago, and fractal geometry hasn't 'looked back' since. Although the notion of a 'fractal dimension' appeared to me at first to be 'over the top', the more I learned about it and used it, the more interested I became. I am very glad to have had the opportunity to do this essay, and it has made a great impact on me.

Finally, I would like to dedicate this work (if I may) to the memory of Felix Hausdorff, one of many victims of the Nazi regime in Germany – I can't understand how such a man could be considered to be a member of the 'lesser race' [16].

[16] *In 1942 he could no longer avoid being sent to the internment camp and, together with his wife and his wife's sister, Felix Hausdorff committed suicide*

6. Bibliography

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