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DIFERENCIJALNE JEDNAČINE SA PERIODIČNIM KOEFICIJENTIMA*

DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

ABSTRACT

The aim of this paper is to explore in some detail the second order linear ordinary differential equation with real or complex periodic coefficients, also known as the *Hill's equation*, with some emphasis on stability and instability intervals and explore two related self-adjoint eigenvalue problems leading to the two final results which enable us to practically solve problems of this type.

ABSTRAKT

Cilj ovog rada je da detaljno istraži linearnu običnu diferencijalnu jednačinu drugog reda sa realnim ili kompleksnim koeficijentima, takodjer znanu kao *Hill-ova jednačina*, posebno posvećujući pažnju intervalima stabilnosti i nestabilnosti i da istraži dva povezana problema svojstvenih vrijednosti vodeći nas do dva finalna rezultata koji nas osposobljavaju da u praksi rješavamo probleme ovog tipa.

1 Hill's equation theory

1.1 Floquet's theory

Let us firstly consider the known general second order differential equation

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \tag{1.1}$$

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where the coefficients $a_s(x)$ $(s = 0, 1, 2)$ are complex-valued, piecewise continuous and periodic, all with period a, where a is a non-zero real constant. It is hence clear that if $\psi(x)$ is a solution of (1.1), then so is $\psi(x + a)$.

Theorem 1.1 *There exist a non-zero constant ρ and a non-trivial solution $\psi(x)$ of (1.1) such that*

$$\psi(x + a) = \rho\psi(x) \tag{1.2}$$

holds. [1]

Now let us extend this in the following theorem.

Theorem 1.2 *There are linearly independent solutions $\psi_1(x)$ and $\psi_2(x)$ of (1.1) such that either*

$$\psi_1(x) = e^{m_1 x} p_1(x), \qquad \psi_2(x) = e^{m_2 x} p_2(x),$$

where m_1 and m_2 are constants, not always distinct, and $p_1(x)$ and $p_2(x)$ are periodic with period a; or

$$\psi_1(x) = e^{mx} p_1(x), \qquad \psi_2(x) = e^{mx} \left(x p_1(x) + p_2(x) \right),$$

where m is a constant and $p_1(x)$ and $p_2(x)$ are periodic with period a. [2]

The first part of the theorem occurs when there are two linearly independent solutions of (1.1), such that (1.2) holds with either different or same values of ρ, while the second part occurs when there is only one such solution. The solutions ρ_1 and ρ_2, whether distinct or not, are called the *characteristic multipliers* of (1.1), and m_1 and m_2 from Theorem (1.2) are called the *characteristic exponents* of (1.1). The above results and their proofs are known as the *Floquet theory* after G. Floquet.

1.2 Hill's equation

Now we finally come to the Hill's equation, and in this part we explore its properties. The name of Hill's equation is given to the equation

$$\{P(x)y'(x)\}' + Q(x)y(x) = 0 \tag{1.3}$$

where $P(x)$ and $Q(x)$ are real valued and have period a. We also assume that $P(x)$ is continuous and nowhere zero and that $P'(x)$ and $Q(x)$ are piecewise

[1] Proof of this theorem can be found in Eastham[3], section 1.1
[2] Proof of this theorem can be found in Eastham[3], section 1.1

continuous. Clearly, this equation is a special case of (1.1) and it is named after G.W. Hill.
We now again look at the two solutions $\psi_1(x)$ and $\psi_2(x)$ from theorem (1.2), but now we use them on equation (1.3). Let $\phi_1(x)$ and $\phi_2(x)$ be the linearly independent solutions of (1.1), which satisfy the conditions

$$\phi_1(0) = 1, \quad {\phi_1}'(0) = 0; \quad \phi_2(0) = 0, \quad {\phi_2}'(0) = 1. \tag{1.4}$$

By the proof of Theorem 1.1, we have that the characteristic multipliers ρ_1 and ρ_2 in the case of Hill's equation are solutions of the quadratic equation

$$\rho^2 - \{\phi_1(a) + {\phi_2}'(a)\}\rho + 1 = 0, \tag{1.5}$$

and hence we have that the characteristic multipliers satisfy

$$\rho_1\rho_2 = 1. \tag{1.6}$$

The solutions $\phi_1(x)$ and $\phi_2(x)$ of (1.3) which satisfy the boundary conditions (1.4) are real valued, by definition of Hill's equation.

Definition 1.3 *The real number D defined by*

$$D = \phi_1(a) + {\phi_2}'(a) \tag{1.7}$$

is called the discriminant of (1.3).

There are five cases we should consider in finding $\psi_1(x)$ and $\psi_2(x)$.

1. $D > 2$. Then

$$\psi_1(x) = e^{mx}p_1(x), \qquad \psi_2(x) = e^{-mx}p_2(x),$$

where $p_1(x)$ and $p_2(x)$ have period a and m is a non-zero real number, by the first part of Theorem 1.2 [3].

2. $D < -2$. Here the situation is the same as in the first case, only m must be replaced by $m + \frac{\pi i}{a}$.

3. $-2 < D < 2$. By (1.5) ρ_1 and ρ_2 are non-real and distinct. Hence by (1.6), and by the fact they are complex conjugates, there exists a real number α with $0 < a\alpha < \pi$ such that

$$e^{ia\alpha} = \rho_1, \qquad e^{-ia\alpha} = \rho_2$$

Then, by Theorem 1.2

$$\psi_1(x) = e^{i\alpha x}p_1(x), \qquad \psi_2(x) = e^{-i\alpha x}p_2(x)$$

where $p_1(x)$ and $p_2(x)$ have period a.

[3] For detailed proofs of all these five results, refer to Eastham [3], Section 1.2

4. $D = 2$. Now we have to decide which part of (1.2) we must apply, because $\rho_1 = \rho_2 = 1$, so we have to consider two cases.

   (a) $\phi_2(a) = \phi_1'(a) = 0$. A simple calculation and a manipulation of the Wronskian [4] of the matrix determined by ϕ_1 and ϕ_2, yields

   $$\psi_1(x) = p_1(x), \qquad \psi_2(x) = p_2(x)$$

   where $p_1(x)$ and $p_2(x)$ have period a. All solutions of (1.3) have period a in this case.

   (b) $\phi_2(a)$ and $\phi_1'(a)$ are not both zero. Here

   $$\psi_1(x) = p_1(x), \qquad \psi_2(x) = xp_1(x) + p_2(x)$$

   where $p_1(x)$ and $p_2(x)$ have period a.

5. $D = -2$. Now $\rho_1 = \rho_2 = -1$, and again as in the previous part we have to consider two cases, depending on the part of Theorem (1.2).

   (a) $\phi_2(a) = \phi_1'(a) = 0$. Doing similar manipulations to the previous part, we get that

   $$\psi_1(x) = e^{\frac{\pi i x}{a}} p_1(x), \qquad \psi_2(x) = e^{\frac{\pi i x}{a}} p_2(x)$$

   where $p_1(x)$ and $p_2(x)$ have period a. In this case all solutions of (1.3) satisfy

   $$\psi(x + a) = -\psi(x)$$

   Let us at this point also note that all functions that satisfy the above conditions are said to be *semi-periodic* with semi-period a.

   (b) $\phi_2(a)$ and $\phi_1'(a)$ are not both zero. Here

   $$\psi_1(x) = P_1(x), \qquad \psi_2(x) = xP_1(x) + P_2(x)$$

   where $P_k(x) = e^{\frac{\pi i x}{a}} p_k(x), (k = 1, 2)$. So obviously, as above, $P_k(x)$ are also semi-periodic.

6. D non real. This is a special case, where D is still defined like in (1.7), only now takes complex values. In this case ρ_1 and ρ_2 are non-real and distinct, and they cannot have modulus unity, because then D

[4] For the Liouville's formula for the Wronskian of two solutions of (1.1), refer to Eastham [2], section 2.3, pages 32-4

would not have complex value, so there is a non-real number m with the property that re $m \neq 0$, such that

$$e^{am} = \rho_1 \qquad e^{-am} = \rho_2$$

So we obtain that

$$\psi_1(x) = e^{mx} p_1(x), \qquad \psi_2(x) = e^{-mx} p_2(x).$$

1.3 Boundedness and periodicity of solutions

Theorem 1.4 *1. If $|D| > 2$, all non-trivial solutions of (1.3) are unbounded in $(-\infty, \infty)$.*

2. If $|D| < 2$, all solutions of (1.3) are bounded in $(-\infty, \infty)$.

This result clearly follows from the cases 1-5 of the value of the discriminant in section 1.2.

Definition 1.5 *The equation (1.3) is said to be*

- *unstable if all non-trivial solutions are unbounded in $(-\infty, \infty)$.*
- *conditionally stable if there is a non-trivial solution which is bounded in $(-\infty, \infty)$.*
- *stable if all solutions are bounded in $(-\infty, \infty)$.*

By Theorem 1.4, (1.3) is unstable if $|D| > 2$, and stable if $|D| < 2$. Periodic and semi-periodic functions are bounded in $(-\infty, \infty)$, so from cases 4 and 5 from section 1.2, we get the following theorem.

Theorem 1.6 *The equation (1.3) has non-trivial solutions with period a if and only if $D = 2$, and with semi-period a if and only if $D = -2$. Moreover, all solutions of (1.3) have period a or semi-period a if and only if $\phi_2(a) = {\phi_1}'(a) = 0$.*

2 Stability and Instability Intervals

We start by extending the definitions of the previous introductory section to a more specific case.

2.1 Extending the previous information

We are still looking at Hill's equation (1.3), but in a slightly more particular form, where $Q(x)$ now has a parameter λ, such that

$$Q(x) = \lambda s(x) - q(x)$$

Here $s(x)$ and $q(x)$ are piecewise continuous with period a and $s(x)$ is bounded from below in the sense that there exists a constant $s > 0$, such that $s(x) \geq s$. Also, if we substitute $P(x)$ with $p(x)$, (1.3) now becomes

$$((p(x)y'(x))' + (\lambda s(x) - q(x))\, y(x) = 0 \tag{2.1}$$

In general, if the functions in the differential equation not only depend upon the variable x and $y(x)$, but also upon a real or complex parameter λ, then the functions $\phi_i(x)$ which form the solution will also depend upon λ. So in our case, we write $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ for the solutions of our equation (2.1) which satisfy the initial conditions (1.4) [5]. So now we define, corresponding to Definition 1.7 the discriminant

$$D(\lambda) = \phi_1(a, \lambda) + {\phi_2}'(a, \lambda) \tag{2.2}$$

Since for all λ, $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ and their derivatives with respect to x are analytic functions for fixed x, then by Definition 2.2 $D(\lambda)$ is an analytic function of λ. Since $D(\lambda)$ is a continuous function of λ, the values of λ for which $|D(\lambda)| < 2$ form an open set on the real $\lambda - axis$. Since this set can be represented as a union of a countable collection of disjoint open intervals, then based on the results of Theorem 1.4, part (2), we can see that (2.1) is stable when λ is in these intervals. Similarly, when λ is in the intervals in which $|D(\lambda)| > 2$, then (2.1) is unstable. Hence, we can formulate the following definition.

Definition 2.1 • *The above described intervals which form the set of values of λ for which $|D(\lambda)| < 2$ are called the* stability intervals *of (2.1).*

[5] Refer to Eastham [2], section 1.7, page 17

- *The intervals which form the set of values of* λ *for which* $|D(\lambda)| > 2$ *are called the* instability intervals *of (2.1).*
- *The intervals which are formed by the closures of the stability intervals are called* conditional stability intervals *of (2.1)* [6].

Note that if λ is complex, then (2.1) has always unstable solutions, and at the endpoints of these intervals the solutions of (2.1) are in general unstable [7].

2.2 The eigenvalue problems

We are going to be dealing here with two eigenvalue problems related to (2.1) and the interval $[0, a]$, and λ is considered as an eigenvalue parameter. Let us now describe the two self-adjoint eigenvalues problems in detail.

1. The *periodic eigenvalue problem.* This problem consists of the Hill equation (2.1), which is taken to hold in $[0, a]$, and we also have the periodic boundary conditions

$$y(a) = y(0), \qquad y'(a) = y'(0) \tag{2.3}$$

This problem is a self-adjoint problem. We also know that the eigenvalues of a self-adjoint eigenvalue problem are real, so we have no problem with the complexity of λ [8]. So, we deduce that the eigenvalues form a countable set with no finite limit points, and we do this in the way of constructing the Green's function and defining a compact symmetric linear operator in an inner-product space. The inner - product space we are dealing with here is that of continuous functions on $[0, a]$ with the inner product

$$< f_1, f_2 >= \int_0^a f_1(x)\overline{f_2(x)}s(x)dx$$

We shall denote the eigenfunctions by $\psi_n(x)$ and the eigenvalues by λ_n where $n = 0, 1, \ldots$ and the sequence of eigenvalues is non-decreasing and $\lambda_n \to \infty$ as $n \to \infty$. We choose $\psi_n(x)$ to be real valued and to form an orthonormal set over $[0, a]$ with weight function $s(x)$. So we have

$$\int_0^a \psi_m(x)\psi_n(x)s(x)dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2.4}$$

[6] These occur when $|D(\lambda)| \leq 2$

[7] Refer to Magnus [5], section 2.1, page 12

[8] Refer to Eastham [2], Chapters 5.1-5.3, pages 84-91 for more information on self-adjoint problems

By (2.3), we can extend $\psi_n(x)$ to the whole $(-\infty,\infty)$ as continuously differentiable functions with period a. Hence the λ_n are the values of λ for which (2.1) has a non-trivial solution with period a.

2. The *semi-periodic eigenvalue problem.* This problem consists of the Hill equation (2.1), which is taken to hold in $[0,a]$, and we also have the semi-periodic boundary conditions

$$y(a) = -y(0), \qquad y'(a) = -y'(0) \tag{2.5}$$

It is also a self-adjoint problem, but this time we shall denote the eigenfunctions by $\xi_n(x)$ and the eigenvalues by $\mu_n(n = 0,1,\ldots)$. Again the sequence of eigenvalues is non-increasing and $\mu_n \to \infty$ as $n \to \infty$. And as before, but now by (2.5) we can extend $\xi_n(x)$ to the whole $(-\infty,\infty)$ as continuously differentiable functions with semi-period a.

From case (4) from the section 1.2 in the case of periodic eigenvalue problem we can deduce that λ_n are the zeros of the function $D(\lambda) - 2$ and that a given eigenvalue λ_n is a double eigenvalue if and only if

$$\phi_2(a,\lambda_n) = \phi_1'(a,\lambda_n) = 0$$

A similar result follows from case (5) from section 1.2 for μ_n, only this time the eigenvalues are the zeros of the function $D(\lambda) + 2$.
From now on, let $\mathcal{F}$ denote the set of all complex-valued functions $f(x)$ which are continuous in $[0,a]$ and have a piecewise continuous derivative in $[0,a]$. Let us now define the Dirichlet integral.

Definition 2.2 *Let $f(x)$ and $g(x)$ be in $\mathcal{F}$. Then the Dirichlet integral $J(f,g)$ is defined to be*

$$J(f,g) = \int_0^a \left(p(x)f'(x)\overline{g'(x)} + q(x)f(x)\overline{g(x)}\right) dx \tag{2.6}$$

If $f(x)$ and $g(x)$ satisfy the boundary conditions (2.3) and if $g(x) = \psi_n(x)$, we get that

$$J(f,\psi_n) = \lambda_n f_n \tag{2.7}$$

where f_n denotes the Fourier coefficient $\int_0^a f(x)\psi_n(x)s(x)dx$, where we have used the fact that $\psi_n(x)$ satisfies (2.1) with $\lambda = \lambda_n$. From equation (2.4) in the periodic eigenvalue problem, we can now deduce that in this case

$$J(\psi_m,\psi_n) = \begin{cases} \lambda_n & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2.8}$$

Theorem 2.3 *Let $f(x)$ be in $\mathcal{F}$ and let it satisfy the boundary conditions (2.3). Then with the Fourier coefficients f_n defined as above, we have that*

$$\sum_{n=0}^{\infty} \lambda_n |f_n|^2 \leq J(f,f)^9. \tag{2.9}$$

Theorem 2.4 *Let $\lambda_{1,n}(n \geq 0)$ denote the eigenvalues in the periodic problem over $[0,a]$. In the problem we replace $p(x)$, $q(x)$ and $s(x)$ by $p_1(x)$, $q_1(x)$ and $s_1(x)$ respectively, where*

$$p_1(x) \geq p(x), \qquad q_1(x) \geq q(x), \qquad s_1(x) \leq s(x) \tag{2.10}$$

Then
(i) *if $s_1(x) = s(x)$ a.e. we have $\lambda_{1,n} \geq \lambda_n$ for all n;*
(ii) *otherwise, we have $\lambda_{1,n} \geq \lambda_n$ provided n is such that $\lambda_n \geq 0$.*

Proof. Let $\psi_{1,n}$ denote the eigenfunction corresponding to the eigenvalue $\lambda_{1,n}$ and let $J_1(f,g)$ denote the Dirichlet integral (2.6) but with $p(x)$ and $q(x)$ replaced by $p_1(x)$ and $q_1(x)$. By (2.10) we have that

$$J_1(f,f) \geq J(f,f) \tag{2.11}$$

Here we prove the theorem for the case 0. So now we consider $f(x) = \psi_{1,0}(x)$. Then by theorem (2.4) we have that

$$\lambda_{1,0} = J_1(\psi_{1,0}, \psi_{1,0}) \geq J(\psi_{1,0}, \psi_{1,0}) \geq \lambda_0 \int_0^a {\psi^2}_{1,0}(x)s(x)dx \tag{2.12}$$

Now by (2.10) we get

$$\int_0^a {\psi^2}_{1,0}(x)s(x)dx \geq \int_0^a {\psi^2}_{1,0}(x)s_1(x)dx = 1$$

Here equality holds in the case (i) of the theorem, while in the second part of the theorem we have strict inequality. Hence, $\lambda_{1,0} \geq \lambda_0$ in the first case, but it only gives $\lambda_{1,0} \geq \lambda_0$ in the second case if $\lambda_0 \geq 0$. This proves the theorem for $n = 0$. [10].

Example 2.5 $p(x) = s(x) = 1$, $q(x) = 0$. *This is an example where (2.1) is reduced to*

$$y''(x) + \lambda y(x) = 0,$$

[9] For proof of this theorem, please refer to [3], section 2.2, page 22
[10] For the rest of the proof please see [3], section 2.2, pages 23-25

a well - known equation. We can show that we have $\lambda_0 = 0$*, and for* $m \geq 0$

$$\lambda_{2m+1} = \lambda_{2m+2} = 4\,(m+1)^2\,\frac{\pi^2}{a^2}$$

$$\mu_{2m} = \mu_{2m+1} = (2m+1)^2\,\frac{\pi^2}{a^2}$$

Example 2.6 $p(x) = 1$, $q(x) = 0$

$$s(x) = \begin{cases} 1 & for\ \left(-\frac{1}{2}a < x \leq 0\right) \\ 9 & for\ \left(0 < x \leq -\frac{1}{2}a\right) \end{cases}$$

The results for the periodic eigenvalue problem are

$$\lambda_{4m+1} = 4\left(m\pi + \frac{1}{2}\alpha\right)^2/a^2, \qquad \lambda_{4m+2} = 4\left((m+1)\pi + \frac{1}{2}\alpha\right)^2/a^2,$$

$$\lambda_{4m+3} = \lambda_{4m+4} = 4(m+1)^2\pi^2/a^2$$

where $\alpha = cos^{-1}\left(\frac{7}{8}\right)$ *and* $0 < \alpha < \frac{1}{2}\pi$*.*

On the other hand, the solution for the semi-periodic eigenvalues problem is

$$\mu_{4m} = 4\left(m\pi + \frac{1}{2}\beta\right)^2/a^2, \qquad \mu_{4m+1} = 4\left(m\pi + \frac{1}{2}\gamma\right)^2/a^2,$$

$$\mu_{4m+2} = 4\left((m+1)\pi - \frac{1}{2}\gamma\right)^2/a^2, \qquad \mu_{4m+3} = 4\left((m+1)\pi - \frac{1}{2}\beta\right)^2/a^2,$$

where $\beta = cos^{-1}\left(\frac{1+\sqrt{33}}{16}\right)$ and $\gamma = cos^{-1}\left(\frac{1-\sqrt{33}}{16}\right)$ and $0 < \beta < \gamma < \pi$.

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